IL
NUOVO CIMENTO
ORGANO DELLA SOCIETÀ ITALIANA DI FISICA
SOTTO GLI AUSPICI DEL CONSIGLIO NAZIONALE DELLE RICERCHE

VOL. XIX, N. 3 *Serie decima* 1° Febbraio 1961

Auger Rate in μ-Mesic Atoms (*).

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(ricevuto il 21 Maggio 1960)

Summary. — A study has been made of the K Auger transitions in the light elements of nuclear emulsions. Of 382 stopping μ^--mesons, 1016 were observed to decay. 5 of these had associated short-range electrons whose energies were consistent with the K transitions of CNO. 2 of these 5 are attributed to the K Auger transitions of CNO, while the remaining 3 are attributed to higher Auger transitions in AgBr. These results are to be compared with the calculations of Burbidge and de Borde which predict 1.3 K Auger electrons from CNO for this experiment. These experimental results are inconsistent with the assumption that the missing K radiative transitions in the experiment of Stearns and Stearns are due to competing Auger processes, since this would require 294 observed K electrons as compared to the 2 actually observed. The experimental results quoted here are in agreement with an earlier experiment of Fry.

The mechanisms involved in the cascade of μ-mesons from an initial atomic state to the ground state have been studied in a number of ways. The relatively long life-time ([1,2]) for decay of the μ-meson, together with the weak interaction for nuclear capture ([2,3]), implies that only atomic phenomena are

(*) This research was supported in part by the Office of Scientific Research, A.R.D.C. of the United States Air Force.

([1]) W. E. BELL and E. P. HINCKS: *Phys. Rev.*, **84**, 1243 (1951).

([2]) J. C. SENS: *Phys. Rev.*, **113**, 679 (1959).

([3]) T. K. TICHO: *Phys. Rev.*, **74**, 1337 (1948).

involved ([4-6]). The transition probabilities for Auger and radiative transitions have been calculated by DE BORDE ([7]) and BURBIDGE and DE BORDE ([8]), from which one can deduce the radiative yield by assuming a particular initial state distribution. A number of experiments have been carried out to ascertain the radiative yield in ligh elements where the Auger effect may be important. The work of STEARNS and STEARNS ([9-12]) especially has indicated a decreasing radiative yield with decreasing atomic number. Since then RUDERMANN ([13]) and FERRELL ([14]) have indicated that the early predictions of BURBIDGE and DE BORDE are essentially correct and that a serious discrepancy exists, since theoretical expectation indicates a very small Auger yield in the light elements. This work was therefore undertaken to ascertain the Auger yield for the μ-mesic cascade in the light elements that exist in emulsions.

The Auger effect in nuclear emulsions exposed to μ-mesons has been studied previously by various workers ([15-17]). The purpose of the present study was to search specifically for the Auger electrons associated with the K transitions in the light elements. The nuclear emulsion is well adapted for this by reason of its composition.

The two groups of elements present in the emulsion, the HCNO group and the AgBr group, are sharply separated in mass and atomic number. The lowest « Bohr » orbits of the μ^--level system of the heavier element group lie so close to the nucleus ([6]) that in spite of the low cross-section for interaction of the μ with nuclear matter, all but approximately 4.5 percent of the μ's trapped in this group are captured by the nucleus ([2-6]). The remainder decay into electrons. In the lighter elements, however, the orbits lie so far outside the nucleus that the majority decay. This means that most of those μ^-'s which show a decay electron at the end of their tracks have been captured by the HCNO group and have cascaded to the lower mesonic levels. Further,

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([4]) E. FERMI and E. TELLER: *Phys. Rev.*, **72**, 399 (1947).
([5]) A. S. WIGHTMAN: *Phys. Rev.*, **77**, 521 (1950).
([6]) D. WEST: *Reports on Progress in Physics*, **21**, 271 (1958).
([7]) A. H. DE BORDE: *Proc. Phys. Soc.*, A **67**, 57 (1954).
([8]) G. E. BURBIDGE and A. H. DE BORDE: *Phys. Rev.*, **89**, 189 (1953).
([9]) M. B. STEARNS and M. STEARNS: *Phys. Rev.*, **105**, 1573 (1957).
([10]) T. DAY and P. MORRISON: *Phys. Rev.*, **107**, 912 (1957).
([11]) N. KRALL and E. GERJUOY: *Phys. Rev. Lett.*, **3**, 142 (1959).
([12]) J. BERNSTEIN and TA-YOU WU: *Phys. Rev. Lett.*, **2**, 404 (1959).
([13]) M. A. RUDERMAN: *X-ray yields from μ-mesonic atoms* (preprin).
([14]) R. A. FERRELL: *Phys. Rev. Lett.*, **4**, 425 (1960).
([15]) M. G. E. COSYNS, C. C. DILWORTH, G. P. S. OCCHIALINI, M. SCHOENBERG and P. PAGE: *Proc. Phys. Soc.*, A **62**, 801 (1949).
([16]) W. F. FRY: *Phys. Rev.*, **79**, 893 (1950); **83**, 594 (1951); **85**, 676 (1952); *Nuovo Cimento*, **10**, 490 (1953).
([17]) C. FRANZINETTI: *Phil. Mag.*, **41**, 86 (1950).

it is believed that the (proton-μ^-) complex, arising from capture of the μ^- by hydrogen, diffuses through the material for a sufficiently long time to result in capture of the μ^- by a heavier element ([18,19]). Effectively, then, no decay takes place in the hydrogen.

Column 3 of Table I gives the distribution, as derived below, of 3382 stopping μ^--mesons amongst the various elements of the emulsion. Column 4 then lists the number expected to decay in each element as obtained from column 3, using the observed lifetimes of reference ([2]). The result is then corrected for geometrical efficiency of detection to give, in column 5, the distribution of the experimentally observed 1016 μ^--decay in this experiment. If

TABLE I.

1	2	3 (a)	4 (b)	5 (b)	6	7	8
Element	Energy of K_α line (keV) (a)	Calculated atomic μ capture distribution	Calculated no. of μ^- decaying in each element	Expected no. of observed decays in each element	Radiative yield [9]	Speculated dist. of observed Auger electrons from column 6 [9]	Calculated no. of observed K electrons from theory [7,8]
Ag	—	988	40	31	—	—	—
Br	—	988	49	39	—	—	—
C	76.6	738	679	535	0.60 ± 0.04	214	.892
N	104.3	169	14β	113	$0.82 \pm .07$	20	.169
O	136.2	499	379	289	$0.80 \pm .05$	60	.220
		3382	1290	1016		294	1.281

(a) Calculated on the basis of a point nucleus.

(b) The data of references ([2]), SENS, and ([22]), SENS, SWANSON, TELEGDI and YOVANOVITCH were used to correct both for absorptions in CNO and μ-decay in AgBr.

A geometrical factor for dip angle bias is applied here to the actual number, 1016, of μ^- observed to decay.

it is presumed that the missing K-level photons in the Stearns and Stearns experiment are all due to Auger competition, then the expected number of Auger electrons accompanying these decays in emulsion is given in column 7.

([18]) L. W. ALVAREZ, H. BRADNER, F. S. CRAWOFRD jr., J. A. CRAWFORD, P. FALKVAIRANT, M. L. GOOD, J. D. GOW, A. H. ROSENFELD, F. T. SOLMITZ, M. L. STEVENSON, H. K. TICHO and R. D. TRIPP: *Phys. Rev.*, **105**, 1727 (1957).

([19]) A. ASHMORE, R. NORDHAGEN, K. STRAUCH and B. M. TOWNES: *Proc. Phys. Soc.* **71**, 161 (1958).

Column 8 lists the corresponding number of Auger electrons based on the Burbidge and de Borde calculations.

This experiment used 600 μm Ilford G-5 plates exposed to the μ^--beam at the University of Chicago synchrocyclotron. The plates were scanned under 275× magnification with all events checked under 1250×. Mesons stopping closer than 10 μm to either surface were not included in the experiment.

A total of 3497 meson endings were finally accepted for the study. Of these, 1016 were observed to decay. These form the basis of the experiment. The results of MORINAGA and FRY ([20]) and of ADELMANN and JONES ([21]) relative to the number of μ^- and π^- giving rise to stars in emulsion were used to determine the π^- contamination in the fiducial region. The total number of μ^- track endings is thus calculated to be 3382.

The total number of decays in the fiducial region, 1290, is obtained from the observed dip-angle distribution of decay electrons by assuming the true distribution to be isotropic. The total number of decays is the sum of the number of decays in each element of the emulsion. The number of μ^--decays in a given element is obtained from the fractional decay rate in that element. This fractional rate is calculated from the results of the lifetime measurement of SENS ([2]).

There is experimental evidence ([22,23]) that in some cases one should assume an atomic capture distribution proportional to the relative abundance of these various elements present and in other cases proportional to the atomic number of the elements. This ambiguity affects both the original distribution and the redistribution of the μ^- from the (proton-μ^-) complexes. For the latter alternate modes are also possible. Calculations based on the various possible combinations of assumptions indicate that the results are insensitive to variations in the modes of distribution and redistribution. The results in Table I have been calculated on the assumptions that the initial atomic capture is proportional to the relative abundance of the elements present, except that no assumption is made as to the capture probability in the heavy elements relative to the light; all the μ^- from the (proton-μ^-) complex are recaptured in the gelatine; the redistribution amongst the elements of the gelatine is proportional to the relative abundances of CNO.

With these assumptions, using the total number of stopping μ^-, 3382; the total number of decays; the known relative abundances; and the decay

([20]) M. MORINAGA and W. F. FRY: *Nuovo Cimento*, **10**, 308 (1953).

([21]) F. ADELMANN and S. JONES: *Phys. Rev.*, **75**, 1468 (1949).

([22]) J. C. SENS, R. A. SWANSON, V. L. TELEGDI and D. D. YOVANOVITCH: *Nuovo Cimento*, **7**, 536 (1958).

([23]) J. F. LATHROP, R. A. LUNDY, R. A. SWANSON, V. L. TELEGDI and D. D. YOVANOVITCH: *Nuovo Cimento*, **15**, 831 (1960).

rates in the various elements as determined above, one obtains a set of simultaneous equations that permits the determination of the distribution of the stopping μ^--mesons amongst the various elements as given in column 3 of Table I. 58% stop in the AgBr and 42% in the gelatine.

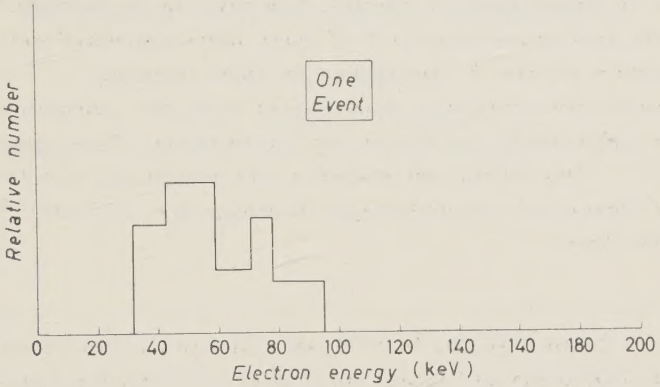

Fig. 1. – Spectrum of observed low-energy electrons greater than 30 keV in association with μ^- decays in emulsion. Based on 1016 observed μ^- decays in emulsion.

The results of the experiment are summarized in Table II. The expermental spectrum of low energy electrons consistent with the energies of K_α Auger electrons from the light elements and appearing in association with a stopping μ^- which decays with emission of an energetic electron is given in Fig. 1. A 25% uncertainty in range due to straggling is included.

TABLE II.

1. Number of stopping μ^-	3382
2. Number of μ-e decays observed	1016
3. Total number of μ-e decays (geometrical correction for bias applied)	1290
4. Percent stopping μ^- which decay	38%
5. μ-e + low energy electron of $30 \leqslant E \leqslant 200$ keV observed	5
6. Number of events under no. 5 attributable to μ^- capture in AgBr	3
7. Number of events in no. 5 due to capture in light elements	2

Each of the five events included in the spectrum was examined under 2500 × magnification and the length of the low energy electron track measured by at least two observers. The energy was determined from the composite range-energy curve for electrons of Demers ([24]). The acceptability cri-

([24]) P. DEMERS: *Ionographie* (Montreal, 1958), p. 250.

terion of COSYNS ([15]) was applied to all apparent Augers in the proper range.

70 of the observed μ-e decays occurred in the AgBr. We expect three of these to be associated with an electron in the energy interval between 30 keV and 200 keV on the basis of the number of electrons observed in this energy region connected with μ^--mesons that stopped without decaying.

Therefore of 1016 observed decays, 946 are due to capture in the light elements CNO, and approximately 2 of these have associated low energy electrons consistent with the K_α transitions in these elements.

These results are completely inconsistent with the presumption that the low radiative yield results are due to the Auger effect. This agrees with Fry's earlier work ([16]). Due to the low statistics, the results can not be said to corroborate the theoretical calculations of Burbidge and de Borde, but they are consistent with them.

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We are very gratefyl to Dr. D. S. HASKIN and to the late Professor MARCEL SCHEIN of the University of Chicago for making available the plates used in this experiment, and wish to thank Dr. V. L. TELEGDI and the University of Chicago Cyclotron group for their co-operation and help.

RIASSUNTO (*)

Si è effettuato uno studio delle transizioni K di Auger negli elementi leggeri delle emulsioni nucleari. Su 3382 mesoni μ^- che si arrestano, se ne osservano 1016 che decadono. 5 di questi sono associati a elettroni a breve percorso le cui energie erano in accordo con le transizioni K del CNO. 2 di queste 5 vengono attribuite a transizioni K di Auger del CNO, mentre le rimanenti 3 sono attribuite a transizioni K di Auger nell'AgBr. Questi risultati vanno confrontati con i calcoli di Burbidge e de Borde che per questo esperimento predicono 1.3 elettroni K di Auger formati nel CNO. Questi risultati sperimentali sono in contrapposto con l'ipotesi che le transizioni K radiative mancanti negli esperimenti di Stearns e Stearns siano dovute a contrastanti processi di Auger, poichè questa ipotesi richiederebbe che si osservassero 294 elettroni K invece dei 2 effettivamente osservati. I risultati sperimentali qui esposti sono in accordo con un experimento precedente di Fry.

(*) *Traduzione a cura della Redazione.*

On Meteor Ablation in the Atmosphere.

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(ricevuto il 1° Luglio 1960)

Summary. — The usually adopted theory of the meteoric ablation is critically examined and improved. More approximate expressions are given for the fundamental equations of the ablation particularly useful for low velocity meteors. New methods are proposed for determining the effective latent heat of ablation and the meteoric density. The results obtained by using these methods shall be published in a following paper.

Introduction.

After the second World War and principally in the last twelve years, the meteoric effects have been extensively investigated by many Authors. However, the meteoric ablation can not yet be described by taking into account all the details of the involved processes. Consequently, the current views are somewhere different.

Recently, important reviews of meteor physics have been made by Öpik [1] and Astapovitch [2]. Pioneer contributions of lasting value to the knowledge of the processes, which the meteors in the terrestrial atmosphere can give rise to, were made by Sparrow [3] and chiefly by Öpik [4,5]; the quantitative theory

[1] E. J. Öpik: *Physics of Meteor Flight in the Atmosphere*, (New York, 1958).

[2] I. S. Astapovitch: *Meteoric Problems in the Earth's Atmosphere*, (in Russian) (Moscow, 1958).

[3] C. M. Sparrow: *Astrophys. Journ.*, **63**, 90 (1926).

[4] E. J. Öpik: *Acta et Comment. Univ. Tartu*, A **33** (1937); or *Tartu Obs. Publ.*, **29**, no. 5 (1937).

[5] E. J. Öpik: *Acta et Comment. Univ. Tartu*, A **26**, no. 2 (1933); or Harvard Coll. Obs. Reprint, no. 100 (1933).

of meteoric ablation was given by HOPPE ([6]) and WHIPPLE ([7,8]). More recently, important works on the subject are due to COOK, EYRING and THOMAS ([9]), THOMAS ([10,11]), THOMAS and WHIPPLE ([12]) and JACCHIA ([13]), besides many studies regarding particular points of the theory.

But a very useful basis for the investigators in a number of problems in pure and applied researches about meteors has been established by Herlofson's equations deduced from a simple model of meteoric ablation ([14]).

The purposes of the present work consist in improving Herlofson's model by eliminating some too particular assumptions and in finding, for fundamental ablation equations an approximation more accurate than that presently used: then, with the aid of some resulting equations, new means are found for determining the ratio of the latent heat of ablation to the heat transfer coefficient, the shape factor and the density of meteors. With some experimental data published by WHIPPLE ([8,15]), these quantities have been evaluated: a following paper will be dedicated to this matter.

1. – The basic hypotheses of ablation theory.

Herlofson's theory is based on the following assumptions:

1) The meteors make individual impacts with single air molecules.

2) All these impacts are completely inelastic and central: then a thermal energy equal to $\frac{1}{2}mv^2$ is produced at the meteor surface at each collision (m, mass of the air molecule; v, meteor speed) and the kinetic energy of the meteor decreases by the quantity mv^2. Accordingly, the molecule receives a momentum mv along the direction of motion. For these evaluations the air molecules have been considered at rest because their velocities are very small (of the order of 100 m/s) if compared to the meteoric velocity v.

3) The meteor ablates before a significant amount of heat flows inward.

4) The energy directly radiated from the meteoric body is negligible.

([6]) J. HOPPE: *Astron. Nach.*, **262**, 169 (1937).
([7]) F. L. WHIPPLE: *Proc. Am. Phil. Soc.*, **79**, 499 (1938).
([8]) F. L. WHIPPLE: *Rev. Mod. Phys.*, **15**, 246 (1943).
([9]) M. A. COOK, H. EYRING and R. N. THOMAS: *Astrophys. Journ.* **113**, 475 (1951).
([10]) R. N. THOMAS: Harvard Reprint Series II-34 (1950).
([11]) R. N. THOMAS: *Astrophys. Journ.*, **116**, 203 (1952).
([12]) R. N. THOMAS and F. L. WHIPPLE: *Astrophys. Journ.*, **114**, 448 (1951).
([13]) L. G. JACCHIA: *Astrophys. Journ.*, **121**, 521 (1955).
([14]) N. HERLOFSON: *Phys. Soc. Rep. Progr. Phys.*, **11**, 444 (1948).
([15]) F. L. WHIPPLE: *Astrophys. Journ.*, **59**, 201 (1954).

5) The meteors are spherical and maintain the spherical shape while evaporating.

6) The atmosphere is in hydrostatic equilibrium, with constant gravity.

The first assumption requires the meteoric dimensions to be small compared with the mean free path of the air molecules at the heights where meteors usually ablate: this condition is satisfied by all the meteors except the brilliant fireballs. The case of brilliant fireballs will be shortly examined.

The second assumption does not seem justified. In this connection it is to be noted that HOPPE ([6]), WHIPPLE ([7-8]), ÖPIK ([1-4]) and many others (see later) assume in their treatment that the meteoric impacts with the air molecules are not completely inelastic, or that a percentage of the impacts are completely inelastic and the others completely elastic. Of course, when an elastic central collision occurs, the meteor loses an amount of energy equal to $2mv^2$, but no heat is produced.

In a not completely inelastic collision of the meteor with an air molecule, the heat Q produced is $Q = \frac{1}{2}A'mv^2$ being A' the coefficient of accommodation, depending upon the value and direction of the final velocity of the molecule. This energy differs from the heat energy $\frac{1}{2}Amv^2$ effectively supplied to the meteoric body for its ablation, by the amount spent on impact radiation, sputtering energy and excitation or ionization of the reflected air molecule, but this difference is presumably small and, when the impacts are single, it isn't necessary to distinguish the coefficient of accommodation A' from the heat transfer coefficient A.

In such a collision the molecule receives a momentum the component of which, along the direction of motion, is γmv. The drag coefficient γ is a dimensionless constant depending by the final velocity of the air molecule.

The air molecules, which collide with a meteoric body element, the norm of which forms an angle α with the direction of flight, will be reflected in different directions: ÖPIK ([1]) assumes that the average direction is that of specular reflection: in this case, if we indicate with e the coefficient of restitution of the impact we easily have $A' = 1 - e^2$; $\gamma = 1 + e \cos 2\alpha$.

As is well known, the order of magnitude of the velocity is the same for all the meteors (from about 10 to about 70 km/s); therefore, it seems acceptable to assume that the restitution coefficient is constant for all the impacts between meteoroid and molecules. For a spherical meteor, the probability that a molecule collides with a surface element, the norm of which forms an angle comprised between α and $\alpha + d\alpha$ with the line of flight is $\sin 2\alpha \, d\alpha$; then with Öpik's assumption, the average momentum received by the molecule in flight direction is simply mv ($\gamma = 1$). It can easily be shown that the same result is also reached when the meteoroid is not spherical but the surface elements have a random distribution ([1]).

Öpik's assumption doesn't satisfy completely: we will write the ablation's equations without making any particular assumption about A and γ, even if presumably A and γ differ only slightly from $(1 - e^2)$ and 1 respectively, in the case of direct single collisions between meteoroid and molecules.

However, in considering larger meteors (fireballs), we know that an hydrodynamic cushion of air is formed in front of the meteor, preventing direct impact of the air molecules on the meteoroid. Under these conditions, A may differ remarkably from A' and heat transfer efficiency is reduced. In this case γ also is smaller than 1 and the atmospheric drag on the meteor decreases. The ablation's equations are the same for general meteors and for fireballs: only the values of A and γ are different. A and γ will be different for every collision. Taking into account the very large number of collisions, we will indicate, in the following with A and γ their average values. Presumably a relation will exist between A and γ, at least in the case of single impacts, because A and γ may be expressed in terms of the elasticity e of the impact. For example, HOPPE ([6]) in his old theory supposed, as ÖPIK, the validity of reflection law and moreover the independence of the recoil velocity of the molecule from the incidence angle: in these conditions, for a spherical meteor, one has: $A = \frac{1}{2}(1 - e^2)$ and $\gamma = \frac{1}{4}(3 + e^2)$ *i.e.* $\gamma = 1 - (A/2)$.

The values assumed by different Authors for the heat transfer coefficient are very different. SPARROW ([3]) quoted $A = 0.27$ to 0.37 while ÖPIK ([4]) in his earlier work, preferred $A = 0.6$ on the basis of the experimental results of VAN VOORHIS and COMPTON ([16]) on the accommodation coefficients of positive ions of argon, neon and helium in their impact with a metallic surface. HOPPE ([6]) from his mass-velocity relation, using Rosenhagen's data, deduced $A = 0.023$, but because of other considerations chose $A = 0.005$. WHIPPLE in his important review ([8]) of 1943 used $A = 0.5$. More recently KAISER ([17]) has assumed $A = 0.7$; THOMAS and WHIPPLE ([12]) and THOMAS ([18]) have regarded $A = 0.1$ as a probable value of the accommodation coefficient; COOK ([19]) by using a modification of the theory of Cook, Eyring and Thomas ([9]) and using the experimental data of a very light meteor with a smooth light curve has deduced $A = 0.77$.

CEPLECHA and HANSEN (see BAKER ([20])) assume respectively $A = 0.54$ and $A = 0.90$. Not to mention others, ÖPIK ([1]) in his recent book uses for stone meteors $A = 0.8$ and LOVELL ([21]) writes in his review that the actual value of A lies between 0.7 and 1.

([16]) C. C. VAN VOORHIS and K. T. COMPTON: *Phys. Rev.*, **37**, 1596 (1931).
([17]) T. R. KAISER: *Adv. Phys.*, **2**, 495 (1953)
([18]) R. N. THOMAS: « *Meteor* » *Special Suppl.*, **2**; to *Journ. Atm. Terr. Phys.*, **1** (1955).
([19]) A. F. COOK: *Astrophys. Journ.*, **120**, 572 (1954).
([20]) R. M. L. BAKER jr.: *Astrophys. Journ.*, **129**, 826 (1959).
([21]) A. C. B. LOVELL: *Handb. d. Phys.*, **48**; *Geophys.*, **2** (1957).

The laboratory experiments with ultra-speed pellets (THOMAS and WHIPPLE [12]) lead to a value $\simeq 0.01$, but the conditions are quite different: the velocities are lower by one order of magnitude (~ 2 km/s) and moreover the pressure is extremely higher.

Some direct information on this quantity is clearly needed.

Likewise, even for what concerns γ, the current opinions are not well defined. WHIPPLE wrote on this subject in his review [8] that « for photographic and visual meteors with which we are concerned the value of γ must be high, not far from unity. For much more massive fireballs, which are not destroyed in the upper atmosphere, smaller values of γ probably are applicable » and later « From the above discussion it is apparent that γ might well be decreased to perhaps 0.9 or 0.8 ». Successively a more profound study on the drag coefficient was carried out by THOMAS [10] and a summary was given also by THOMAS and WHIPPLE [12]. They wrote: « Since, at the highest meteor velocities and lowest air densities, we expect most of the air molecules to be absorbed by the meteor surface, a value of γ between $\frac{1}{2}$ and 1 would seem the most reasonable » and later « In summary, for spherically shaped meteors, we expect γ to be about $\frac{1}{2}$, with an uncertainty factor of about 2, depending on air density and speed. We might also expect some decrease in γ as we go from the faster meteors in the upper atmosphere to the slower meteors in the lower ».

Very recently BAKER [20] has applied the theory of transitional aerodynamic drag to meteors and has proposed a simple expression for the variation of γ « more realistic than the almost universally employed constant of 1 ». According to this Author, the molecules emitted from the meteoric body have an important effect upon the drag and cannot be neglected, as is usually done.

The third and fourth assumptions can be substantially accepted. Researches recently carried out by THOMAS [11,18] have supported the idea that heat flow doesn't affect the ablation of meteors. Moreover, if we accept for the meteor surface a temperature value of about 3 000 °K (see, for example, THOMAS [11] and COOK [19]), following ÖPIK's discussion [1], it may be concluded that radiation is insignificant for the meteor ablation. Micrometeorites are obviously excluded from this last consideration.

The assumption that meteors are spherical can be easily eliminated by introducing a shape factor. Let us call $\sigma = \sigma(v)$ the cross-sectional area of the meteoroid; according to HOPPE [6] a shape factor may be introduced, defined as $A = \sigma M^{-\frac{2}{3}}$, where $M = M(v)$ is the mass of the meteor. For a spherical particle $\sigma(v) = \pi r^2(v)$, $M(v) = \frac{4}{3}\pi\varrho r^3(v)$, where ϱ is the mean density of the meteor; then $A = (9\pi/16)^{\frac{1}{3}}\varrho^{-\frac{2}{3}} = 1{,}21\,\varrho^{-\frac{2}{3}}$ turns out to be independent of M and r, being a function of the sole ϱ. Obviously, for meteors other than spherical, A depends also on the shape and orientation of the particle. The proper meaning of A lies in its independence from M, which postulates that σ varies with the dimensions of the meteors as r^2 (r = equivalent radius of the

meteor defined by the relation $M = \frac{4}{3}\pi\varrho r^3$). For a spherical iron meteor ($\varrho = 7.8$ g/cm^3) we have $A = 0.31$ cm$^2 \cdot$g$^{-\frac{2}{3}}$, while for a spherical stone meteor the classical value of ϱ is 3.4 g/cm^3, hence $A = 0.53$ cm$^2 \cdot$g$^{-\frac{2}{3}}$. The value used by WHIPPLE ([8-22]) for A is 0.5 cm$^2 \cdot$g$^{-\frac{2}{3}}$; KAISER ([16]) adopted $A = 0.48$ cm$^2 \cdot$g$^{-\frac{2}{3}}$. JACCHIA ([13]) used $\gamma A = 1$ so that $A \geqslant 1$.

At present, a considerable uncertainty affects the value of the density of meteoric material ([21]): following WHIPPLE ([23,24]) the meteoric density should be 0.05 g/cm^3; accordingly, A should have a value about 9 for a spherical particle. Because of this uncertainty in ϱ, it is perhaps preferable that the shape factor be independent of the density; therefore, in more recent works a dimensionless shape factor $A_0 = A\varrho^{\frac{2}{3}}$ is used (WHIPPLE ([23]), HAWKINS and SOUTHWORTH ([25])). For bodies of simple shape (cones, ellipsoids, short cylinders) A_0 is of the order of unity ([25]) (for a sphere $A_0 = 1.21$). ÖPIK ([1]) defines a shape parameter as the dimensionless quantity $B = rS/V$ (S, area of the minimum convex envelope of the body, V compact volume of the meteor). For a compact sphere $B = 3$; this is the minimum value B can assume. We will use a shape factor F defined as $\sigma = F\pi r^2$; so that for a sphere, we have $F = 1$. Evidently it is $A_0 = (9\pi/16)^{\frac{1}{3}}F$.

Finally the sixth assumption is the usual scheme of the atmosphere; in a very recent work WEISS ([26]) has considered the case of a non isothermal atmosphere with a linear scale height gradient, but for our purposes, the differences are not essential.

2. – The meteoric ablation equations.

During a time $\mathrm{d}t$ the air intercepted by the meteoric body is $\mathrm{d}m = \varrho_a \sigma \,\mathrm{d}s$, being ϱ_a the air density and $\mathrm{d}s$ the path element; the heat produced is $\mathrm{d}Q = (\Lambda/2)\,\mathrm{d}m\, v^2$ and the energy carried away by the ablated atoms is $-l\,\mathrm{d}M$, where $\mathrm{d}M$ is the mass loss of the meteor and l the latent heat of ablation for unit mass. The conservation laws of energy and momentum are then expressed by

$$\frac{A}{\varrho} v^2 \,\mathrm{d}m = -l\,\mathrm{d}M\,, \tag{1}$$

$$\gamma v \,\mathrm{d}m = -M\,\mathrm{d}v\,. \tag{2}$$

([22]) F. L. WHIPPLE: *Adv. in Geophys.*, **1**, 119 (1952).
([23]) F. L. WHIPPLE: Harvard Reprint 465 (1957).
([24]) F. L. WHIPPLE: *Vistas in Astronautics* (London, 1958), p. 115.
([25]) G. S. HAWKINS and R. B. SOUTHWORTH: Harvard Reprint Series II-128 (1958).
([26]) A. A. WEISS: *Austr. Journ. Phys.*, **12**, 54 (1959).

By dividing equation (1) by equation (2) and putting $\gamma(l/\Lambda) = \xi$ we obtain

$$\frac{1}{2\xi} r\,\mathrm{d}r = \frac{\mathrm{d}M}{M}, \tag{3}$$

which, upon integration, gives

$$M = M_0 \exp\left[\frac{v^2 - v_0^2}{4\xi}\right], \tag{4}$$

if ξ is constant during the ablation, as we will suppose, though recently JACCHIA ([27]) has found in photographic meteors a little increase of ξ during the ablation.

It is worth noting that equation (4) is independent of the shape of the meteor.

By introducing the equivalent radius of the meteor, equation (4) assumes the form

$$r = r_0 \exp\left[\frac{r^2 - v_0^2}{12\xi}\right]. \tag{4'}$$

Taking into account the expressions of $\mathrm{d}m$ and σ and remembering that $\varrho_a = p\,gH$ (p atmospheric pressure, g gravity acceleration, H scale height), equation (2) gives

$$\mathrm{d}p = -\frac{4\varrho g r_0}{3\gamma F v} \cos\chi \exp\left[\frac{v^2 - v_0^2}{12\xi}\right] \mathrm{d}v, \tag{5}$$

where χ, the zenithal angle of the meteoric path in the atmosphere, may be considered constant. In order to integrate eq. (5) it is necessary to known the function $F = F(v)$: as a first approximation, we can assume $F(v) = \text{constant}$, because the meteors, during the flight, undergo oscillations and rotations, due to the impacts with the air molecules ([1]).

After integration eq. (5) becomes

$$p = \frac{2\varrho g r_0}{3\gamma F} \cos\chi \exp\left[\frac{-v_0^2}{12\xi}\right] \cdot \left| E\left(\frac{v_0^2}{12\xi}\right) - E\left(\frac{v^2}{12\xi}\right) \right|, \tag{6}$$

where $E(x)$ indicates the logarithmic integral.

The ablation rate n, *i.e.* the number of atoms ablated per unit time is

$$n = -\frac{1}{\mu} \cdot \frac{\mathrm{d}M}{\mathrm{d}t}, \tag{7}$$

where μ is the mass of an individual meteor atom.

([27]) L. G. JACCHIA: Harvard Reprint Series II-122 (1958).

Substituting eq. (1) into eq. (7) we have

$$n = \frac{\pi\gamma F}{2\mu\xi g H} pr^2 v^3 . \tag{8}$$

Moreover, since the path of the meteor within the earth's atmosphere may be assumed to be a straight line, recalling eq. (3), (7), the modulus a of the deceleration of the meteor is given by

$$a = \frac{3F\gamma}{4\varrho g H} \frac{v^2}{r} p . \tag{9}$$

The explicit expressions of n and a in terns of v are

$$n = \frac{M_0}{4\mu H\xi} \cos\chi \exp\left[\frac{-v_0^2}{4\xi}\right] \cdot v^3 \exp\left[\frac{v^2}{6\xi}\right] \cdot \left[E\left(\frac{v_0^2}{12\xi}\right) - E\left(\frac{v^2}{12\xi}\right)\right] , \tag{8'}$$

$$a = \frac{\cos\chi}{2H} v^2 \exp\left[-\frac{v^2}{12\xi}\right] \cdot \left[E\left(\frac{v_0^2}{12\xi}\right) - E\left(\frac{v^2}{12\xi}\right)\right] . \tag{9'}$$

The electronic line density q of the ionized column produced by the meteor is given by

$$q = \frac{n}{v}\beta , \tag{10}$$

β being the probability that a single evaporated atom will produce a free electron. Recently a power-law dependence of β upon velocity v has been assumed, so that

$$\beta = \beta_0 v^\eta . \tag{11}$$

Evans and Hall ([28]) gave $\eta = 0.5 \pm 0.5$ but the last independent evaluations of η (Whipple ([29]), Hawkins ([30]), Weiss ([31])) indicate a value of about 5. Equations (8), (10) and (11) give

$$q = \frac{\pi F\gamma\beta_0}{2\mu\xi\, gH} pr^2 v^{\eta+2} , \tag{12}$$

([28]) S. Evans and J. E. Hall: « *Meteors* » *Special Suppl.*, **2**; to *Journ. Atmosph. Terr. Phys.*, 18 (1955).

([29]) F. L. Whipple: *Astrophys. Journ.*, **121**, 241 (1955).

([30]) G. S. Hawkins: *Astrophys. Journ.*, **124**, 311 (1956).

([31]) A. A. Weiss: *Austr. Journ. Phys.*, **10**, 397 (1957).

or explicitly

$$q = \frac{M_0\beta_0}{4\mu\xi H}\cos\chi\exp\left[\frac{-v_0^2}{4\xi}\right]\cdot v^{\eta+2}\exp\left[\frac{v^2}{6\xi}\right]\cdot\left[E\left(\frac{v_0^2}{12\xi}\right)-E\left(\frac{v^2}{12\xi}\right)\right]. \tag{12'}$$

According to the current views, the luminous intensity I in the spectral range 4500 Å to 5700 Å is

$$I = -\frac{1}{2}\tau v^2\frac{\mathrm{d}M}{\mathrm{d}t}, \tag{13}$$

τ being the luminous efficiency in the spectral range considered; following WHIPPLE ([8,21]) the luminous efficiency is proportional to the meteoric speed:

$$\tau = \tau_0 v. \tag{14}$$

Then, by using eq. (7) we have

$$I = \frac{\mu\tau_0}{2} n v^3, \tag{15}$$

or explicitly

$$I = \frac{M_0\tau_0}{8H}\cos\chi\exp\left[\frac{-v_0^2}{4\xi}\right] v^6\exp\left[\frac{v^2}{6\xi}\right]\cdot\left[E\left(\frac{v_0^2}{12\xi}\right)-E\left(\frac{v^2}{12\xi}\right)\right]. \tag{15'}$$

3. – The first approximation of the ablation's equations and some of its applications.

A first approximation of the ablation's equations may be obtained when $(v_0 - v)/v_0$ is far below unity during the entire appreciable phase of ablation, as it occurs when the initial speed of the meteor is greater than about (30 ÷ 40) km/s ([32]). With such a condition we have

$$E\left(\frac{v_0^2}{12\xi}\right)-E\left(\frac{v^2}{12\xi}\right)=\int_{v^2/12\xi}^{v_0^2/12\xi}\frac{e^x}{x}\,\mathrm{d}x = \frac{12\xi}{\bar{v}^2}\left[\exp\left[\frac{v_0^2}{12\xi}\right]-\exp\left[v^2/12\xi\right]\right]\simeq \tag{16}$$

$$\simeq \frac{12\xi}{v_0^2}\left[\exp\left[\frac{v_0^2}{12\xi}\right]-\exp\left[\frac{v^2}{12\xi}\right]\right].$$

By using eq. (16), it is easy to find a maximum for the ablation rate and, if we neglect ξ in comparison with v_0^2, the maximum of electronic line density and luminous intensity occur at the same instant and the following approx-

([32]) F. VERNIANI: *Ric. Scient.*, **29**, 1965 (1959).

imate expressions can be reached:

$$n = \frac{9}{4} n_{\max} \frac{p}{p_{\max}} \left(1 - \frac{1}{3} \frac{p}{p_{\max}}\right)^2, \tag{17}$$

$$q = \frac{9}{4} q_{\max} \frac{p}{p_{\max}} \left(1 - \frac{1}{3} \frac{p}{p_{\max}}\right)^2, \tag{18}$$

$$I = \frac{9}{4} I_{\max} \frac{p}{p_{\max}} \left(1 - \frac{1}{3} \frac{p}{p_{\max}}\right)^2, \tag{19}$$

$$v = v_0 \left[1 + \frac{12\xi}{v_0^2} \ln\left(1 - \frac{1}{3} \frac{p}{p_{\max}}\right)\right]^{\frac{1}{2}}, \tag{20}$$

$$a = \frac{2}{3} a_{\max} \frac{p}{p_{\max}} \left(1 - \frac{1}{3} \frac{p}{p_{\max}}\right)^{-1}, \tag{21}$$

$$r = r_0 \left(1 - \frac{1}{3} \frac{p}{p_{\max}}\right), \tag{22}$$

$$M = M_0 \left(1 - \frac{1}{3} \frac{p}{p_{\max}}\right)^3, \tag{22'}$$

$$n_{\max} = \frac{4}{9\mu H} M_0 v_0 \cos\chi, \tag{23}$$

$$q_{\max} = \frac{4}{9\mu H} M_0 \beta_0 v_0^\eta \cos\chi, \tag{24}$$

$$p_{\max} = \frac{8\varrho g \xi}{3\gamma F} \frac{r_0}{v_0^2} \cos\chi, \tag{25}$$

$$z_{\max} = H \ln \frac{3 p_0 \gamma F v_0^2 \sec\chi}{8\varrho g r_0 \xi}, \tag{25'}$$

$$r_{\max} = \frac{2}{3} r_0, \tag{26}$$

$$M_{\max} = \frac{8}{27} M_0, \tag{26'}$$

$$v_{\max} = v_0 - \frac{12\xi}{5 v_0}, \tag{27}$$

$$a_{\max} = \frac{3\xi}{H} \cos\chi, \tag{28}$$

$$I_{\max} = \frac{2\tau_0}{9H} M_0 v_0^4 \cos\chi, \tag{29}$$

where $n_{\max}$, $q_{\max}$ and $I_{\max}$ are respectively the maximum of the ablation rate, of the electronic line density and of the luminous intensity, p_0 is the atmospheric pressure at sea level and $p_{\max}$, $a_{\max}$, $z_{\max}$, $M_{\max}$, $r_{\max}$, $v_{\max}$, are the values of p, a, z, M, r, v at the point where the maxima of n, q and I occur.

Equations (17), (18), (23), (26), (26′) coincide with the Herlofson's equations because they do not contain either the shape factor F or the coefficient of accommodation A or the drag coefficient γ; however, it is to be emphasized that these equations have been found by assuming F independent of v, *i.e.* of the position of the meteor along its path in the atmosphere.

It is important to note that the ablation energy per unit mass l appears always divided by the accommodation coefficient A, so that an experimental separate determination of one of these two quantities is impossible. We can regard this ratio l/A as the effective latent heat of ablation.

From eq. (27) we have

$$\xi = \frac{5}{12} v_0(v_0 - v_{\max}) . \tag{30}$$

In this way ξ can be determined by measuring the velocity at the point of maximum ionization and by evaluating the velocity outside the atmosphere by means of the usual extrapolation.

For the radiometeors, the accuracy, with which the velocity can be measured, is often poor, so that the value of ξ may be remarkably affected by the errors in the values of v_0 and $v_{\max}$.

In every case eq. (28) suggests the possibility of another independent determination of ξ. We have

$$\xi = \frac{H a_{\max}}{3 \cos \chi} . \tag{31}$$

Also the value $a_{\max}$ of the deceleration in the point of maximum ionization is difficult to be found with the required accuracy; then it is expedient to substitute $a_{\max}$ with an average value of a, which is much more easily determined. The average value $\bar{a}$, of the deceleration in the time interval t_1, t_2 corresponding to the heights z_1, z_2 and velocities v_1, v_2 is

$$\bar{a} = \frac{v_1 - v_2}{t_2 - t_1} = \frac{6\xi \cos \chi}{z_1 - z_2} \ln \frac{3p_{\max} - p_1}{3p_{\max} - p_2} . \tag{32}$$

Hence

$$\xi = \frac{\bar{a} \sec \chi}{6} (z_1 - z_2) \left[\ln \frac{3p_{\max} - p_1}{3p_{\max} - p_2} \right]^{-1} . \tag{33}$$

Jointly with equations (18) and (19), equation (32) gives account of the well known fact [33] that fast meteors may have measurable trajectories ex-

[33] L. G. Jacchia and F. L. Whipple: Harvard Reprint Series II-99 (1956).

tending over 50 km or more without any detectable deceleration, while slow meteors with large zenithal angles may show remarkable deceleration.

If the physical process of ablation is postulated (for example, evaporation), the latent heat l may be thermodynamically calculated, and by the experimental value of ξ, the ratio γ/Λ is obtained. Thus, when γ and Λ may be expressed in terms of e, they are separately determined.

In principle, for each meteor, the shape factor F may be deduced by using eq. (25), which then becomes

$$F = \frac{8}{3}\left(\frac{3}{4\pi}\right)^{\frac{1}{3}} \frac{g M_0^{\frac{1}{3}} \varrho^{\frac{2}{3}}}{\gamma v_0^2 p_{\max}} \xi \cos \chi , \tag{34}$$

but for an effective determination of F, the value of ϱ is needed. Because of the actual uncertainty upon the value of ϱ it is now preferable to assume for F an average value and to use eq. (34) in order to determine the order of magnitude of ϱ.

The value of meteoric density cannot be found directly; the experimental data may give only $\varrho r_0/F$ and ϱr_0^3 (see eq. (24) or (29) and (25)): then we have only two equations and three parameters. However ϱ may be determined by measuring the values of $\varrho r_0/F$ and ϱr_0^3 for many meteors, if we suppose that all meteors have the same density. Let us suppose that $\varrho r_0/F$ and ϱr_0^3 have been measured for ν meteors and that $c_1, c_2 \ldots c_i \ldots c_\nu$; $k_1, k_2, \ldots k_i, \ldots k_\nu$ are the results of the measurements:

$$\frac{\varrho r_{01}}{F_1} = c_1; \qquad \frac{\varrho r_{02}}{F_2} = c_2; \quad \ldots \quad \frac{\varrho r_{0i}}{F_i} = c_i; \quad \ldots \quad \frac{\varrho r_{0\nu}}{F_\nu} = c_\nu, \tag{35}$$

$$\varrho r_{01}^3 = k_1; \qquad \varrho r_{02}^3 = k_2; \quad \ldots \quad \varrho r_{0i}^3 = k_i; \quad \ldots \quad \varrho r_{0\nu}^3 = k . \tag{36}$$

The (35) and (36) are 2ν equations with $(2\nu+1)$ unknown quantities; but, if ν is large, we can write

$$\sum_i^{1\ldots\nu} F_i = \nu \tag{37}$$

by assuming that the mean value of the shape factor F for a great number of meteors will be 1.

Then with eq. (35), (36), (37) we obtain

$$\varrho = \nu^{\frac{3}{2}} \left(\sum_1^\nu {}_i\, k_i^{\frac{1}{2}} c_i^{-1}\right)^{-\frac{3}{2}}. \tag{38}$$

This formula has been used by the author to provide a determination of meteoric density: the results, to be published soon, clearly show the existence of different types of meteors.

4. – More approximate expressions for ablation equations.

The approximate equations from (17) to (29) are reached through the following assumptions:

(*a*) $$\overline{v} = v_0 \,, \qquad \text{(see eq. (16))}$$

(*b*) $$\xi \ll v_0^2 \,.$$

WEISS ([34]) has recently improved eqs. (24), (25), (26), (27), by retaining assumption (*a*), but rejecting assumption (*b*). The equations which can substitute eqs. (24), (25), (26), (27), affording a more realistic description are (*)

(39) $$q_{\max} = \frac{12 f^2 \varphi^{-3}}{\mu H} M_0 \beta_0 v_{\max}^{\eta} \cos\chi \,,$$

(40) $$p_{\max} = \frac{8 \varrho g \xi r_0}{\varphi \gamma F v_0^2} \cos\chi \,,$$

(41) $$r_{\max} = \frac{2f}{\varphi} r_0 \,,$$

(42) $$v_{\max} = v_0 \left(1 + \frac{12\xi}{v_0^2} \ln\frac{2f}{\varphi}\right)^{\frac{1}{2}} ,$$

where f and q are functions of v_0, ξ, η defined by

(43) $$f = 1 + \frac{3(2+\eta)\xi}{v_0^2} \,,$$

(43′) $$\varphi = 1 + 2f \,,$$

and the subscript max refers to maximum electronic line density.

Obviously if we neglect in these equations $3(2+\eta)\xi/v_0^2$ in comparison with unity, we reobtain for $q_{\max}$, $p_{\max}$, and $r_{\max}$ the expressions (24), (25), (26). The equations (40), (41), (42) represent an effective improvement in respect to the preceding approximations only for slow meteors (*i.e.* when the term $3(2-\eta)\xi/v_0^2$ is not negligible in comparison with unity).

In the same order of approximation, it is possible to find the equations

([34]) A. A. WEISS: *Austr. Journ. Phys.*, **11**, 591 (1958).

(*) In Weiss' work, the shape factor F and drag coefficient γ are omitted ($F=1$, $\gamma=1$ as in Herlofson's theory).

which can substitute eqs. (18), (19), (20), (21), (22), (28), (29):

$$q = \frac{\varphi^2}{4f^2} q_{\max} \frac{p}{p_{\max}} \left(1 - \frac{p}{\varphi p_{\max}}\right)^2 \left[\frac{1 + (12\xi/v_0^2) \ln (1 - p/\varphi p_{\max})}{1 + (12\xi/v_0^2) \ln (2f/\varphi)}\right]^{(\eta+2)/2}, \tag{44}$$

$$I = \frac{\varphi^2}{4f^2} I_{\max} \frac{p}{p_{\max}} \left(1 - \frac{p}{\varphi p_{\max}}\right)^2 \left[\frac{1 + (12\xi/v_0^2) \ln (1 - p/\varphi p_{\max})}{1 + (12\xi/v_0^2) \ln (2f/\varphi)}\right]^3, \tag{45}$$

$$v = v_0 \left[1 + \frac{12\xi}{v_0^2} \ln \left(1 - \frac{p}{\varphi p_{\max}}\right)\right]^{\frac{1}{2}}, \tag{46}$$

$$u = \frac{2f}{\varphi} a_{\max} \frac{p}{p_{\max}} \left(1 - \frac{p}{\varphi p_{\max}}\right)^{-1} \frac{1 + (12\xi/v_0^2) \ln (1 - p/\varphi p_{\max})}{1 + (12\xi/v_0^2) \ln (2f/\varphi)}, \tag{47}$$

$$r = r_0 \left(1 - \frac{p}{\varphi p_{\max}}\right), \tag{48}$$

$$a_{\max} = \frac{3\xi}{fH} \cos \chi \left(1 + \frac{12\xi}{v_0^2} \ln \frac{2f}{\varphi}\right), \tag{49}$$

$$I_{\max} = \frac{6f^2 \varphi^{-3} \tau_0}{H} M_0 v_{\max}^4 \cos \chi . \tag{50}$$

It must be remarked that now $I_{\max}$ represents only the value of the luminous intensity I at the point at which the electronic line density q is maximum and doesn't coincide with the actual maximum I_m of the same luminous intensity: I_m may be obtained even by eq. (50) if only we put 4 instead of η in eqs. (43) and (43′) which give f and φ. On this matter, it is easy to add that the electronic line density q reaches its maximum before the luminous intensity I, which, in its turn, precedes the maximum of the ablation rate u.

If a preliminar value ξ_0 of the effective heat of ablation ξ is known, by using the Weiss' approximation, we may find a more reliable one. Keeping the same symbolism of the preceding section, we obtain ξ by solving the equation:

$$\bar{u} \quad \frac{v_1 - v_2}{t_2 - t_1} = \frac{v_0}{t_2 - t_1} \left\{\left[1 + \frac{12\xi}{v_0^2} \ln \left(1 - \frac{p_1}{\varphi p_{\max}}\right)\right]^{\frac{1}{2}} - \left[1 + \frac{12\xi}{v_0^2} \ln \left(1 - \frac{p_2}{\varphi p_{\max}}\right)\right]^{\frac{1}{2}}\right\}, \tag{51}$$

where φ is computed through ξ_0.

Equations (44) to (48) are valid for $x \geqslant x_f$, *i.e.* when the condition $p \leqslant q p_{\max}(1 - \exp[x_f - x_0])$ is satisfied, where $x = r^2/12\xi$ and being x_f the lowest value x may assume during the ablation:

$$x_f - x_0 = \frac{1}{3} \ln \frac{\mu}{M_0},$$

when $\frac{1}{3}\ln \mu/M_0$ is greater than about $(0.1 - x_0)$ and $x_f = 0.1$ in the other case ([32]). For meteors slower than about 30 km/s we can always take $x_f = 0.1$ ([32]). For the others the preceding condition practically becomes $p < \varphi p_{max}$, because of $(\mu/M_0)^{\frac{1}{3}}$ is fully negligible in comparison with unity. By concluding equations (44) to (48) are valid for $p < \varphi p_{max} b_1(x_0)$, where the function $b_1(x_0)$ is defined by

$$b_1(x_0) = 1 - \exp(x_f - x_0) = \begin{cases} 1 - \exp[0.1 - x_0], & x_0 < \sim 7.5, \\ 1 - (\mu/M_0)^{\frac{1}{3}} & x_0 > \sim 7.5. \end{cases} \tag{52}$$

Table I contains the values of $b_1(x_0)$ for $\xi = 10^7$ joule/kg: thus we see that for meteors faster than about 25 km/s (more than 90% ([35])), the limit of validity becomes simply $p < \varphi p_{max}$. However, for the little part of meteors with

TABLE I.

v_0 (km/s)	x_0	$f(x_0)$	$\varphi(x_0)$	$b_1(x_0)$	$b(x_0)$	$\varphi b_1(x_0)$	$\varphi b(x_0)$
11	1.008	2.735	6.470	0.597	1.290	3.863	8.346
13.19	1.450	2.207	5.414	0.741	1.619	4.012	8.765
15	1.875	1.933	4.866	0.831	1.758	4.044	8.554
16.47	2.261	1.774	4.548	0.885	1.793	4.025	8.154
20	3.333	1.525	4.050	0.961	1.666	3.892	6.750
25	5.208	1.336	3.672	0.994	1.381	3.650	5.071
30	7.500	1.233	3.466	0.999	1.207	3.464	4.183
35	10.208	1.171	3.342	1.000	1.128	3.342	3.770
40	13.333	1.131	3.262	1.000	1.090	3.262	3.555
45	16.875	1.103	3.206	1.000	1.068	3.206	3.424
50	20.833	1.084	3.168	1.000	1.053	3.168	3.336
55	25.208	1.069	3.138	1.000	1.044	3.138	3.276
60	30.000	1.058	3.116	1.000	1.036	3.116	3.228
65	35.208	1.050	3.099	1.000	1.031	3.099	3.195
70	40.833	1.043	3.086	1.000	1.027	3.086	3.169

very low velocity, the limit is further reduced: for instance, if $v_0 = 15$ km/s, p must be smaller than $\sim 0.83 \varphi p_{max}$. From the values of the Table I, we see that eqs. (44) to (48) have a wider validity field than the corresponding equations of first approximations, which hold only when $p \leqslant 3 p_{max} b_1(x_0)$.

Now we will investigate the degree of accuracy of eqs. (44) to (48) and the relative extension of their validity field. The assumption $\bar{v} = v_0$ is really acceptable when r differs only slightly from r_0, as is well known, also experimentally, to happen when $v \geqslant v_{max}$. However this position becomes unre-

([35]) D. W. R. MCKINLEY: *Astrophys. Journ.*, **113**, 225 (1951).

liable when v ceases to be near v_0. To compare the effective values of the highest atmospheric pressure p_f at which a meteor of given characteristics may arrive during its ablation with the maximum value p_{fa} allowed by the approximate equations, we proceed as follows.

The expression of p_f is given by eq. (6):

$$p_f = k \exp[-x_0]\left[E(x_0) - E(x_f)\right], \tag{53}$$

where, recalling eq. (40)

$$k = \frac{2\varrho g r_0 \cos\chi}{3\gamma F} = \varphi p_{\max} x_0 . \tag{54}$$

Putting

$$b_f(x) = x \exp[-x]\int_{x_f}^{x} \frac{e^u}{u}\,\mathrm{d}u , \tag{55}$$

and

$$b(x) = x \exp[-x]\int_{0,1}^{x} \frac{e^u}{u}\,\mathrm{d}u , \tag{55'}$$

we have

$$p_f = \varphi p_{\max} b_f(x_0) \tag{56}$$

and for meteors slower than about 30 km/s

$$p_f = \varphi p_{\max} b(x_0) , \tag{56'}$$

while for the other meteors

$$p_f < \varphi p_{\max} b(x_0) . \tag{56''}$$

From eq. (52) and (55) it is easy to see that always $b_f(x_0) > b_1(x_0)$; thus we have the inequality

$$\varphi p_{\max} b_1(x_0) < p_f \leqslant \varphi p_{\max} b(x_0) . \tag{57}$$

The function $b(x_0)$ has a maximum for $x_0 = 2.26$, the value of which is 1.79, then it decreases and its limit for $x_0 \to \infty$ is 1. In Table I are given the values of $b(x_0)$, $\varphi b(x_0)$, $\varphi b_1(x_0)$ as functions of x_0, for $\xi = 10^7$ joule/kg and $\eta = 5$. The function $\varphi b(x_0)$ has a maximum for $x_0 = 1.45$ and its value is 8.76; then it decreases and its limit for $x_0 \to \infty$ is 3. $\varphi b_1(x_0)$ has a maximum for $x_0 = 1.87$ and its value is 4.04. Then p_{fa} doesn't overcome $\sim 4 p_{\max}$ while p_f can reach the value of $8.75 p_{\max}$; p_f is more than twice p_{fa} for initial

speeds less than about 16.5 km/s, but for velocities greater than (35 ÷ 40) km/s p_{fa} differs slightly from the effective p_f; this result is not surprising because in these cases v_f is near enough to v_0 ([32]).

In the Fig. 1 to 7, x/x_0 is plotted as a function of p/p_{max} for different values of initial velocities ($v_0 = 11$ km/s; $v_0 = 15$ km/s; $v_0 = 20$ km/s; $v_0 = 30$ km/s; $v_0 = 40$ km/s; $v_0 = 50$ km/s; $v_0 = 60$ km/s): the curve c is obtained by eq. (6)

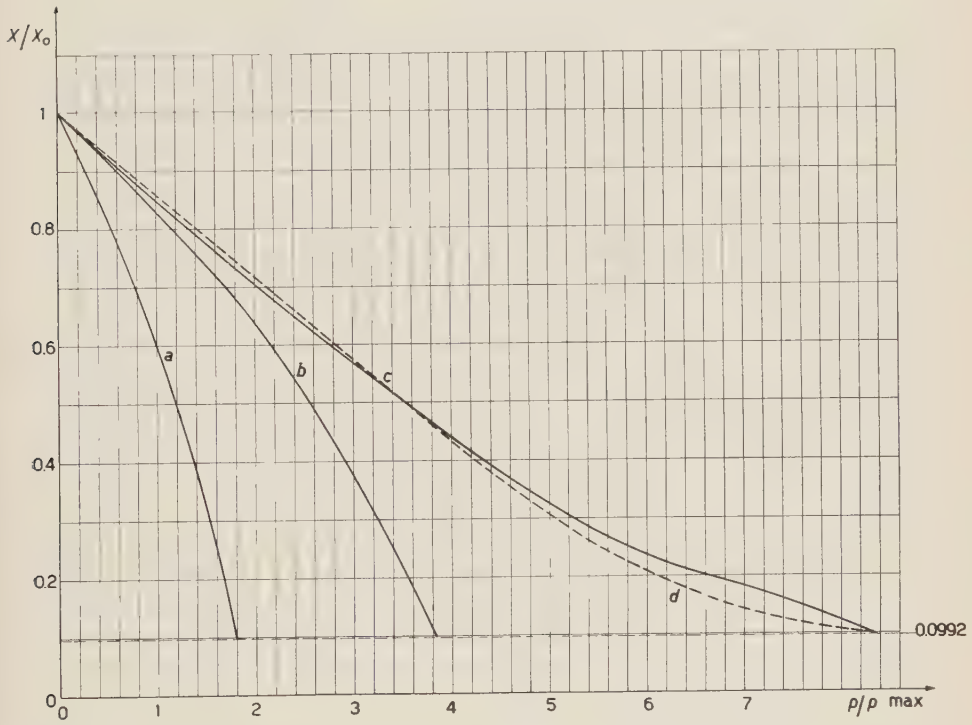

Fig. 1. Comparison of the exact and approximate equations for meteor speed as function of the pressure. $x = v^2/12\xi$; $x_0 = 1.008$ ($v_0 = 11$ km/s). Curve a: first approximation; b: Weiss' approximation; c: exact solution; d: new approximation.

while the curves a, b represent respectively Herlofson's and Weiss' approximations. Conversely curve d shows a new approximation, which will be introduced later. In the Fig. 5, 6, 7 the horizontal straight lines show the values of x_f/x_0 at which the ablation process comes at the end for meteors of different mass.

The electron density profiles, the deceleration and radius of the meteor, as given by the exact solution and the various approximations are compared in Fig. 8, 9 and 10 for three values of the initial speed ($v_0 = 20$ km/s; $v_0 = 40$ km/s; $v_0 = 60$ km/s): the values of q/q_{max} and a/a_{max} as a function of p/p_{max} afforded by Herlofson's approximation are independent of the speed.

It is worth noting that the deceleration of meteors can reach a maximum, when the initial velocity is small or the mass is large; this maximum occurs

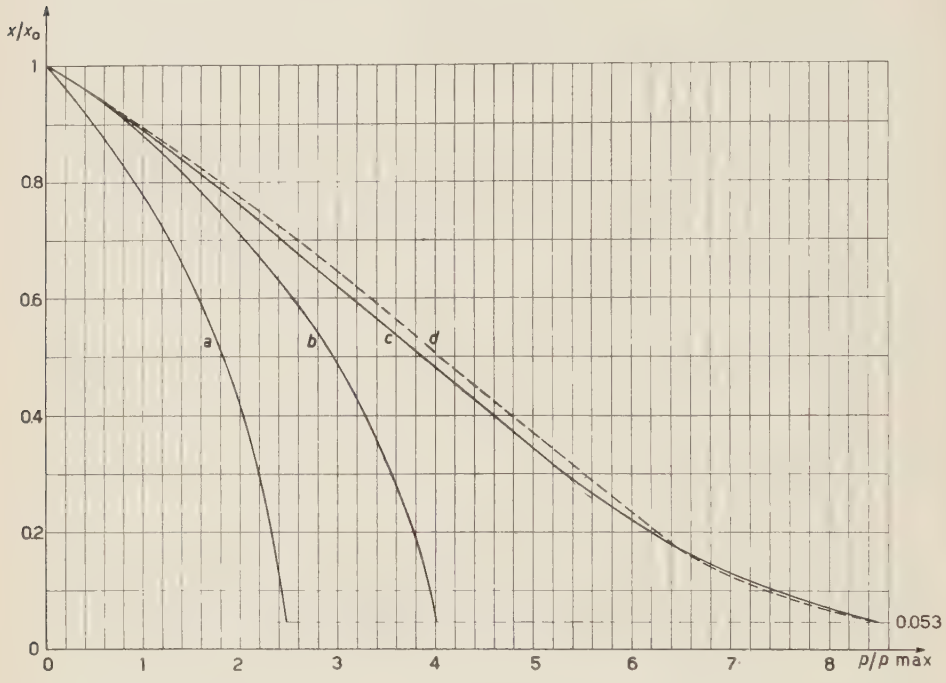

Fig. 2. – Comparison of the exact and approximate equations for meteor speed as function of the pressure. $x=v^2/12\xi$; $x_0=1.875$ ($v_0=15$ km/s). Curve a: first approximation; b: Weiss' approximation; c: exact solution; d: new approximation.

for the values x_m satisfying the equation:

$$e^x = (1-x)\left[E(x_0) - E(x)\right], \tag{58}$$

which requires $x<1$, *i.e.* $v^2<12\,\xi$, $v<\sim 11$ km/s. For a great part of the meteors, v is greater than 11 km/s during the ablation and the deceleration increases until the end of ablation ([32-36]). For the meteors that satisfy (58), the value of maximum deceleration a_m is given by

$$a_m = \frac{6\xi \cos\chi \cdot x_m}{H(1-x_m)}. \tag{59}$$

([36]) F. VERNIANI: *Nuovo Cimento*, **14**, 938 (1959).

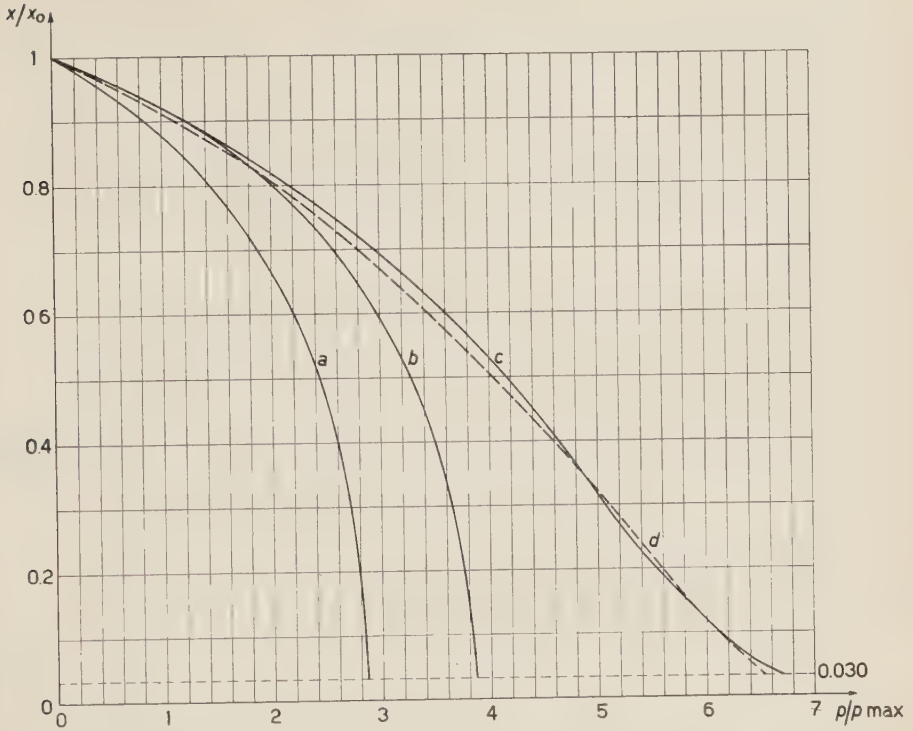

Fig. 3. – Comparison of the exact and approximate equations for meteor speed as function of the pressure. $x=v^2/12\xi$; $x_0=3.333$ ($v_0=20$ km/s). Curve a: first approximation; b: Weiss' approximation; c: exact solution; d: new approximation.

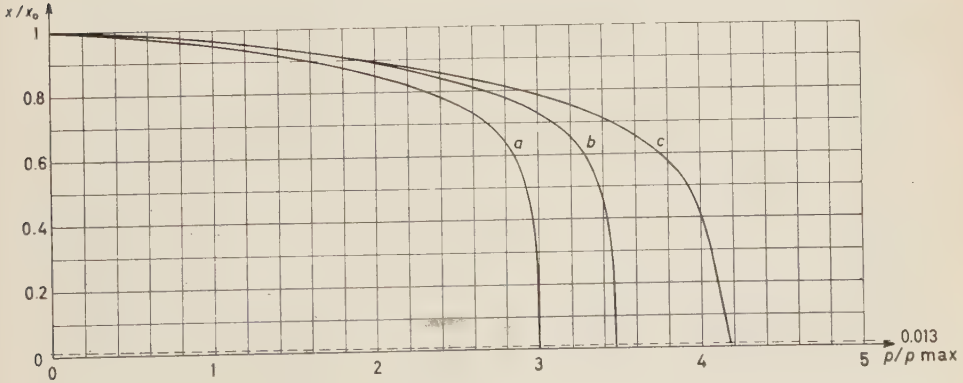

Fig. 4. – Comparison of the exact and approximate equations for meteor speed as function of the pressure. $x=v^2/12\xi$; $x_0=7.500$ ($v_0=30$ km/s). Curve a: first approximation; b: Weiss' approximation; c: exact solution.

By solving numerically eq. (58), it is possible to see that, except for the slowest meteors, the values of x_m are very close to unity: if the initial speed is greater than about 30 km/s, x_m exceeds 0.99.

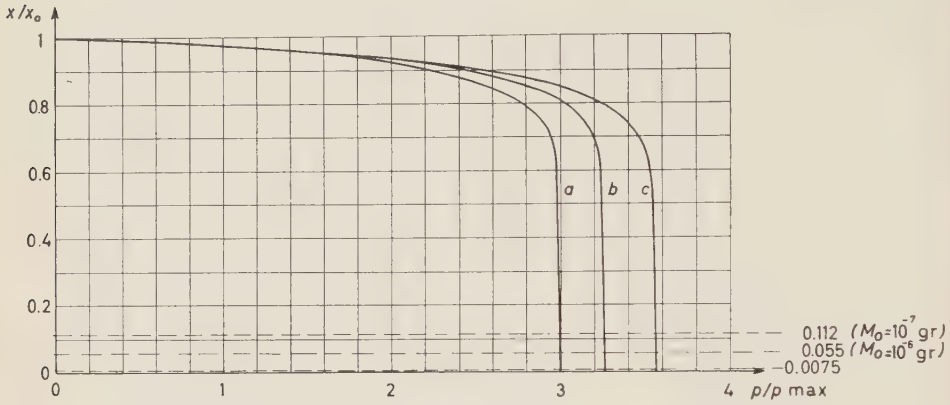

Fig. 5. – Comparison of the exact and approximate equations for meteor speed as function of the pressure. $x=v^2/12\xi$; $x_0=13.333$ ($v_0=40$ km/s). Curve a: first approximation; b: Weiss' approximation; c: exact solution.

From the examination of the plotted curves, we may conclude that, for meteors slower than about 30 km/s, first approximation completely fails. Conversely, Weyss' approximation affords a remarkable improvement, particularly for slow meteors. Therefore it may be accepted that, except for very slow meteors; another approximation is clearly needed in this last case.

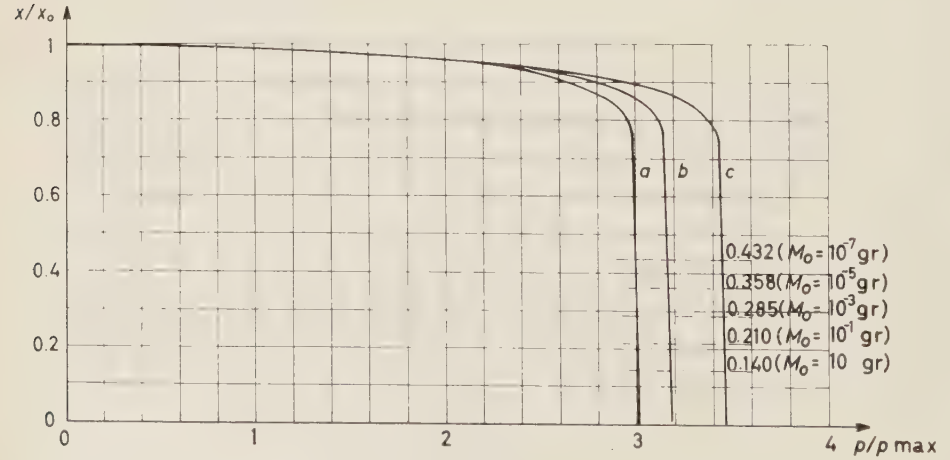

Fig. 6. – Comparison of the exact and approximate equations for meteor speed as function of the pressure. $x=v^2/12\xi$; $x_0=20.833$ ($v_0=50$ km/s). Curve a: first approximation; b: Weiss' approximation; c: exact solution.

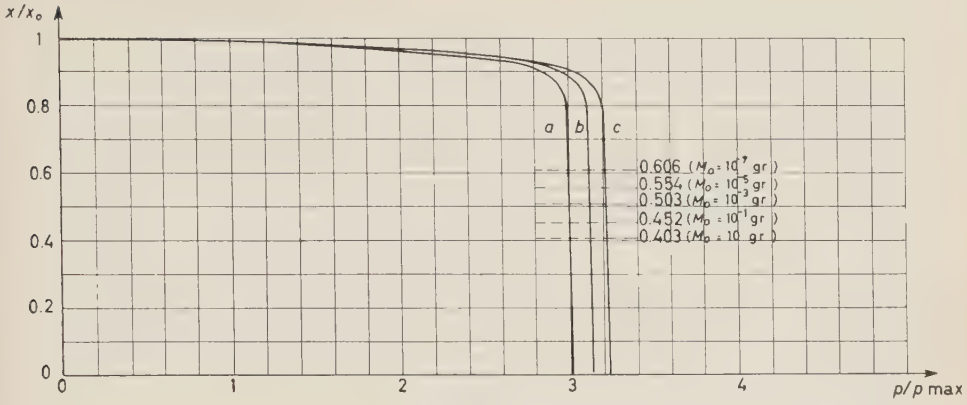

Fig. 7. – Comparison of the exact and approximate equations for meteor speed as function of the pressure; $x=v^2/12\xi$; $x_0=30.000$ ($v_0=60$ km/s). Curve a: first approximation; b: Weiss' approximation; c: exact solution.

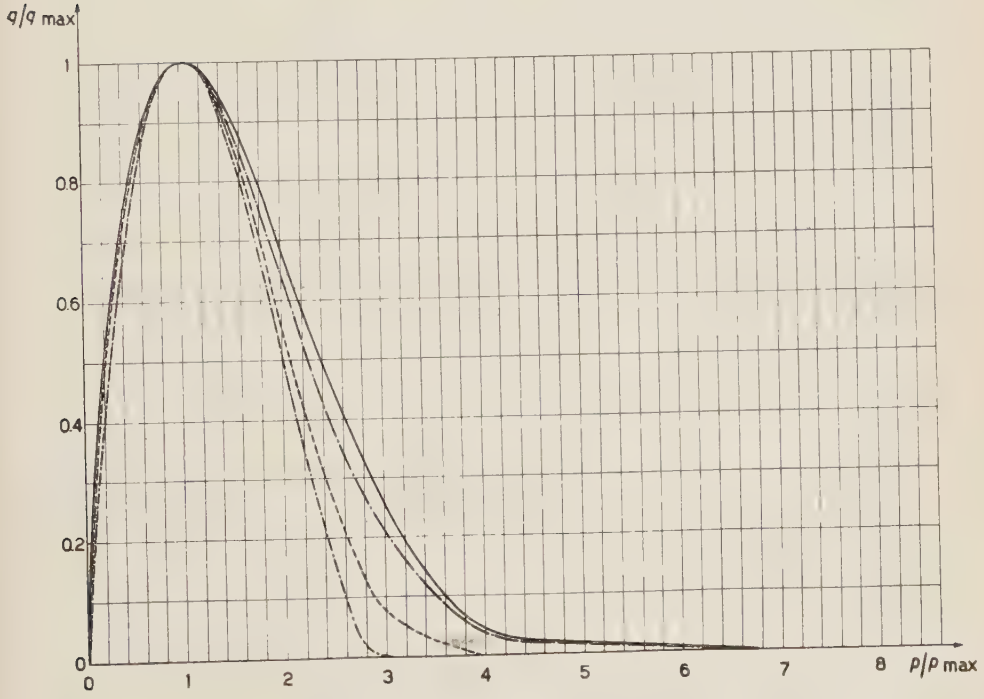

Fig. 8*a*. – Comparison of the exact and approximate equations for electronic line density as function of the pressure. First approximation: –·–·–; Weiss' approximation – – –. Exact solution ——. New approximation: —·—·—. $v_0=20$ km/s.

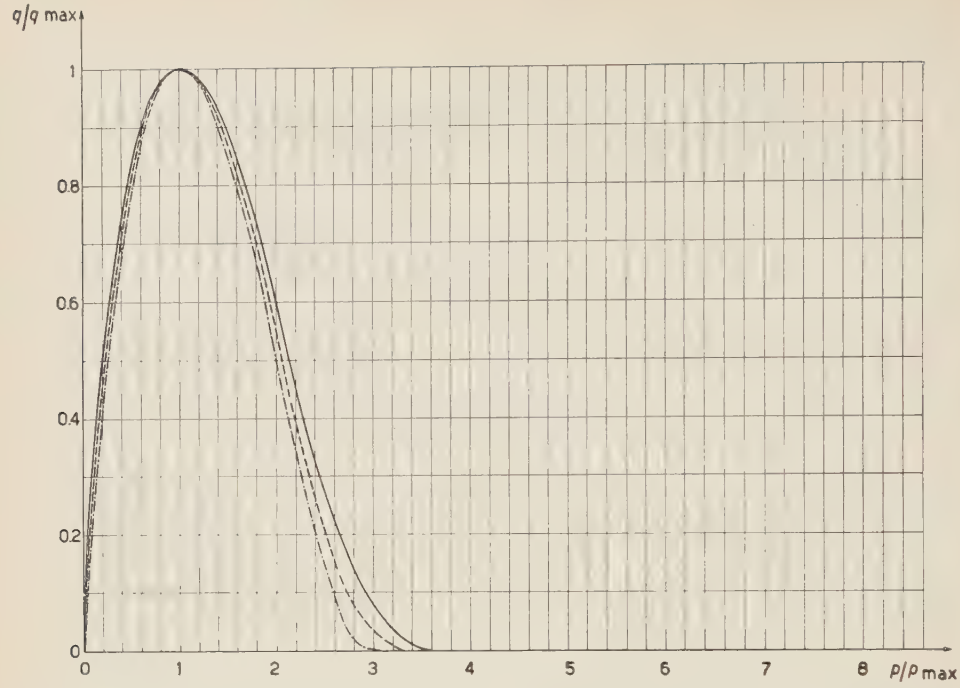

Fig. 8*b*. – Comparison of the exact and approximate equations for electronic line density as function of the pressure. First approximation: –·–·–; Weiss' approximation – – –. Exact solution ———. $v_0 = 40$ km/s.

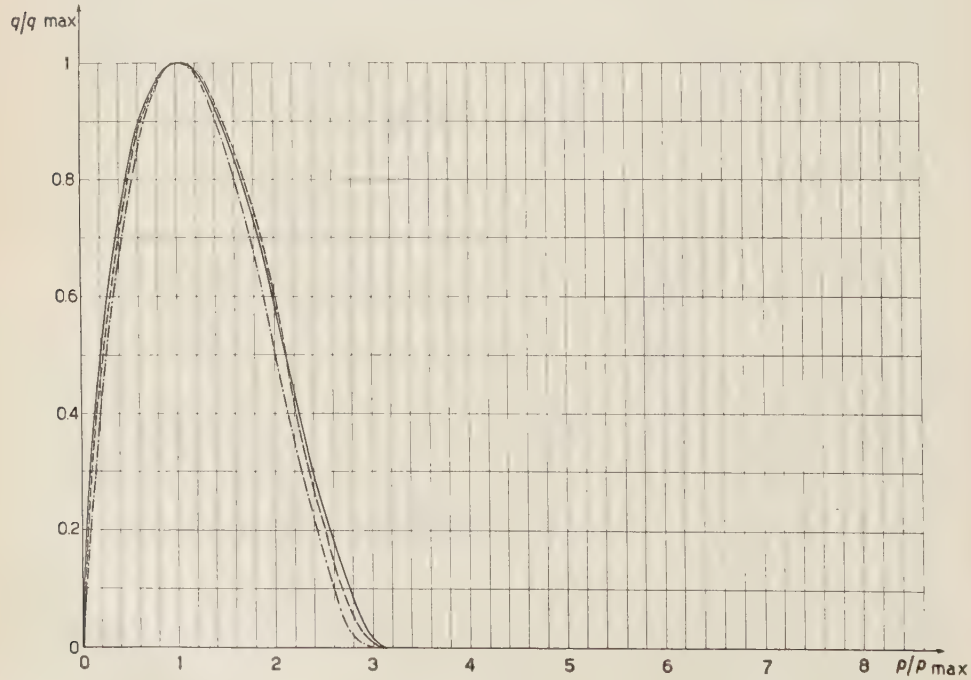

Fig. 8*c*. – Comparison of the exact and approximate equations for electronic line density as function of the pressure. First approximation: –·–·–; Weiss' approximation – – – Exact solution ——— . $v_0 = 60$ km/s.

Recently, at Jodrell Bank ([37]), serious departures from the theoretical height distribution as function of velocity have been experimentally found for

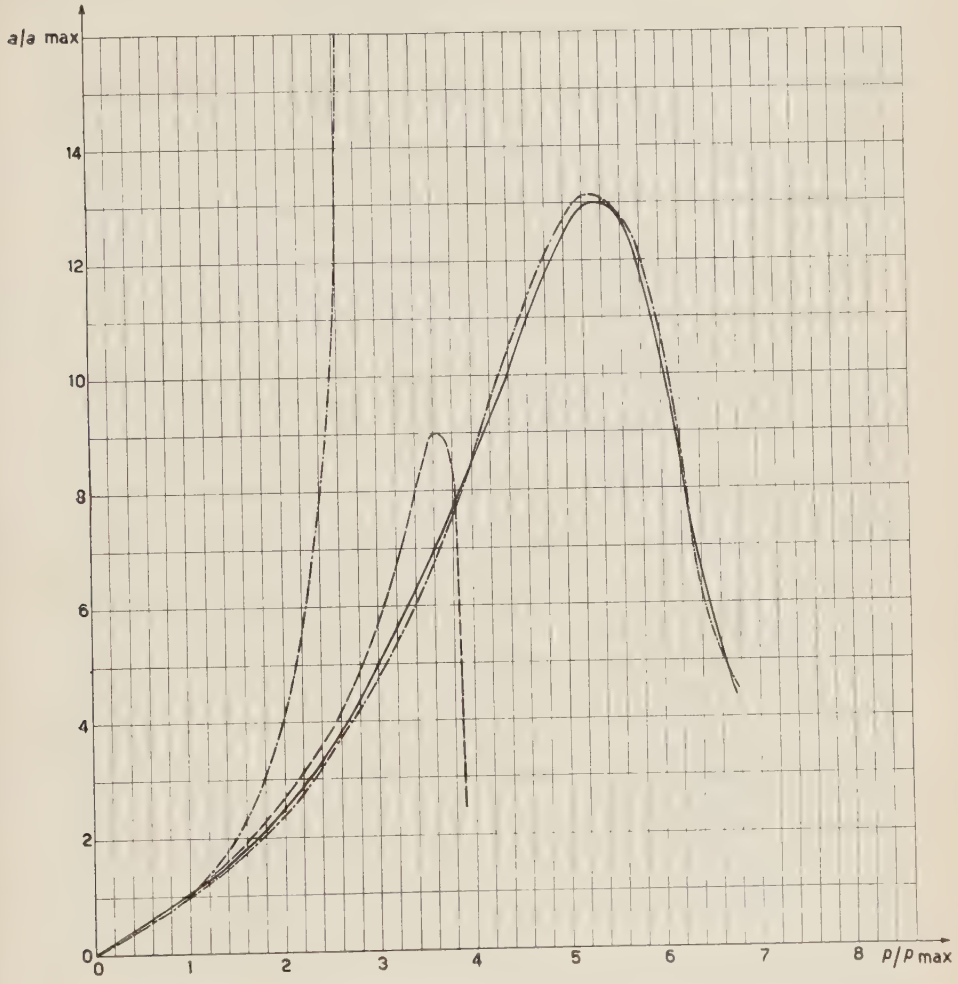

Fig. 9*a*. Comparison of the exact and approximate equations for meteoric deceleration as function of the pressure. First approximation: –·–·–. Weiss' approximation: – – –. Exact solution: ——. New approximation: —·—·—. $v_0 = 20$ km/s.

velocities lower than 30 km/s; while for greater velocities, the agreement between theory and experience is quite good: the very slow meteors appear to ionize at heights where the atmospheric pressure is less than expected. The effect may be easily explained by using eq. (40) instead of eq. (25): following

([37]) Technical (Final) Report, Contract no. AF 61(514)-948, Jodrell Bank (1958).

eq. (40), p_{max} is proportional to

$$\frac{1}{v_0^2(1+14\xi/v_0^2)},$$

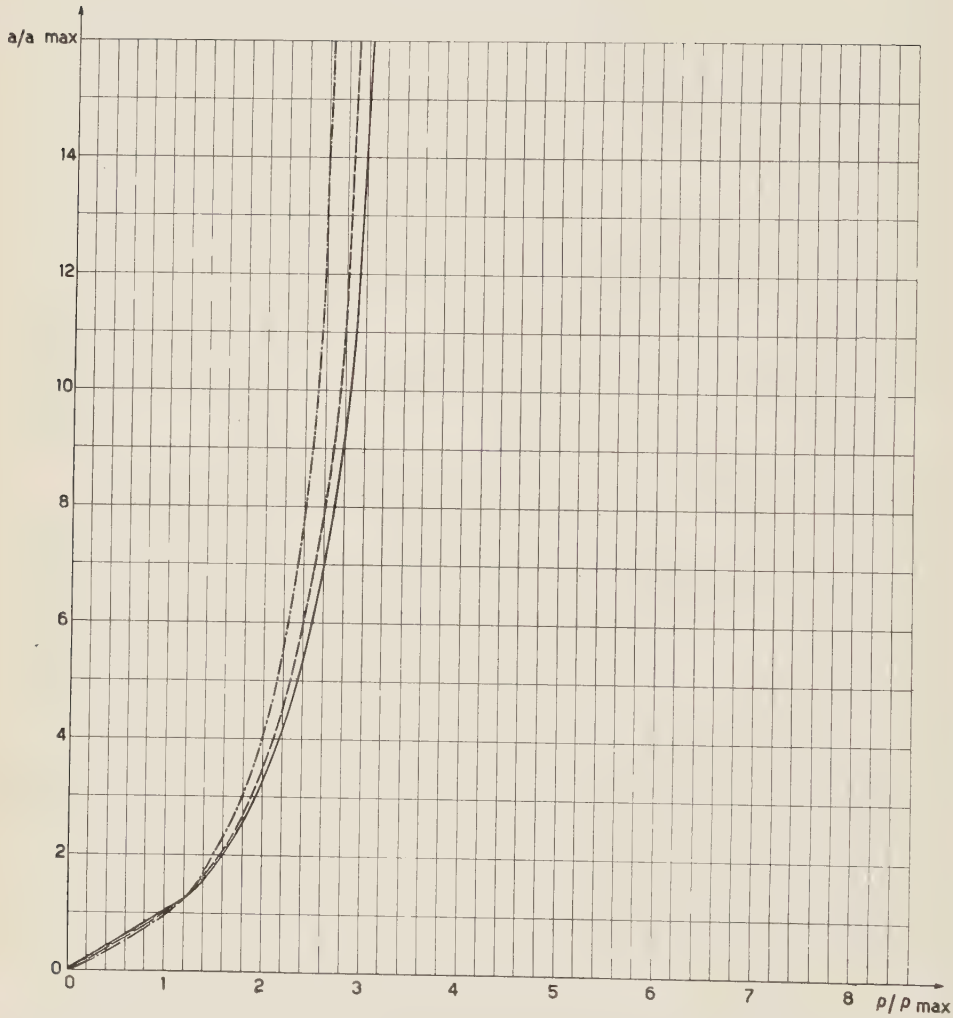

Fig. 9*b*. – Comparison of the exact and approximate equations for meteoric deceleration as function of the puressure. First approximation: –·–·–. Weiss' approximation: – – –. Exact solution: ——. v_0=40 km/s.

instead of simply to v_0^{-2}; the disagreement shown by the Fig. 4 of the cited paper among the experimental points and the theoretical curve is in great part removed.

For meteors having an initial speed less than $(20 \div 25)$ km/s, we propose the following analytical expression in place of (46):

$$\frac{x}{x_0} = 1 + \frac{1}{x_0} \ln \left\{ 1 - A \frac{p}{\varphi p_{\max}} + B \left(\frac{p}{\varphi p_{\max}} \right)^2 \right\}, \tag{60}$$

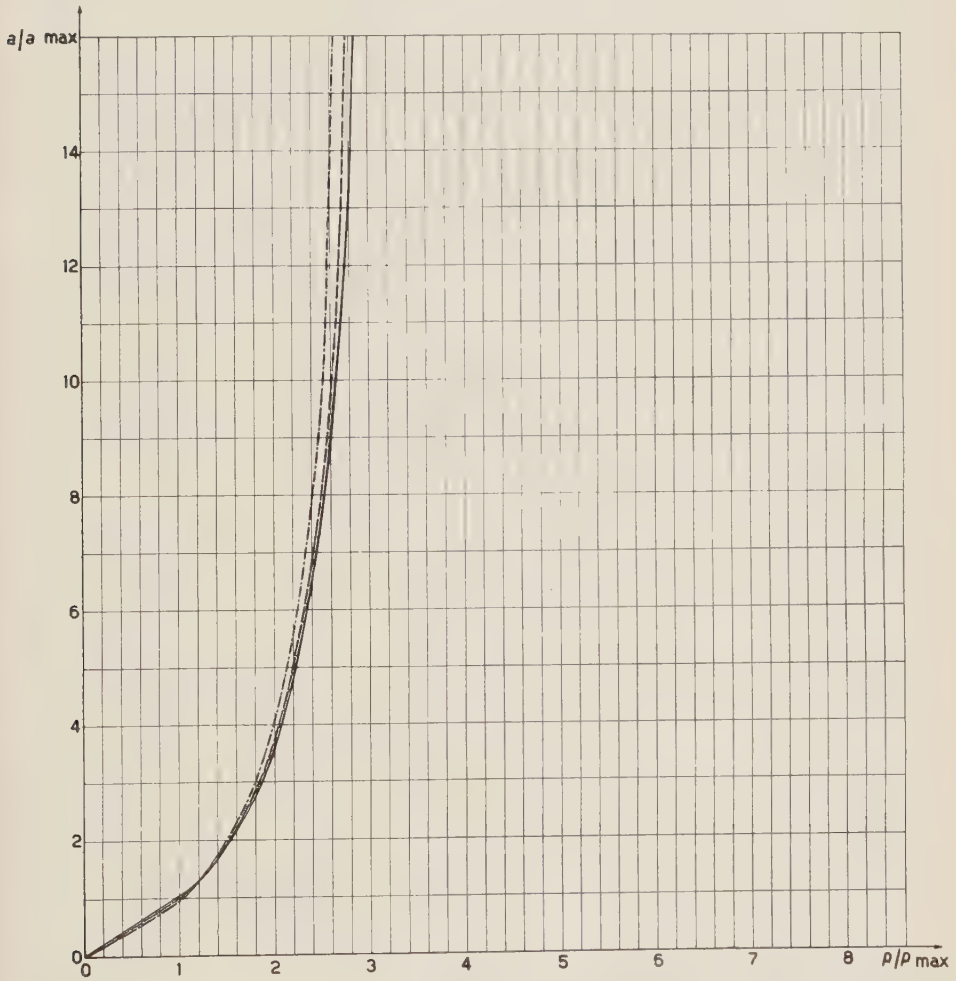

Fig. 9c. – Comparison of the exact and approximate equations for meteoric deceleration as function of the pressure. First approximation: –·–·–. Weiss' approximation: – – –. Exact solution: ———. $v_0 = 60$ km/s.

where A and B can be regarded as roughly constant: the values that appear preferable are $A = 2b_1/b$, $B = b_1/b^2$, where b_1 and b are the functions defined by eq. (52) and (55′), the values of which are given in Table I.

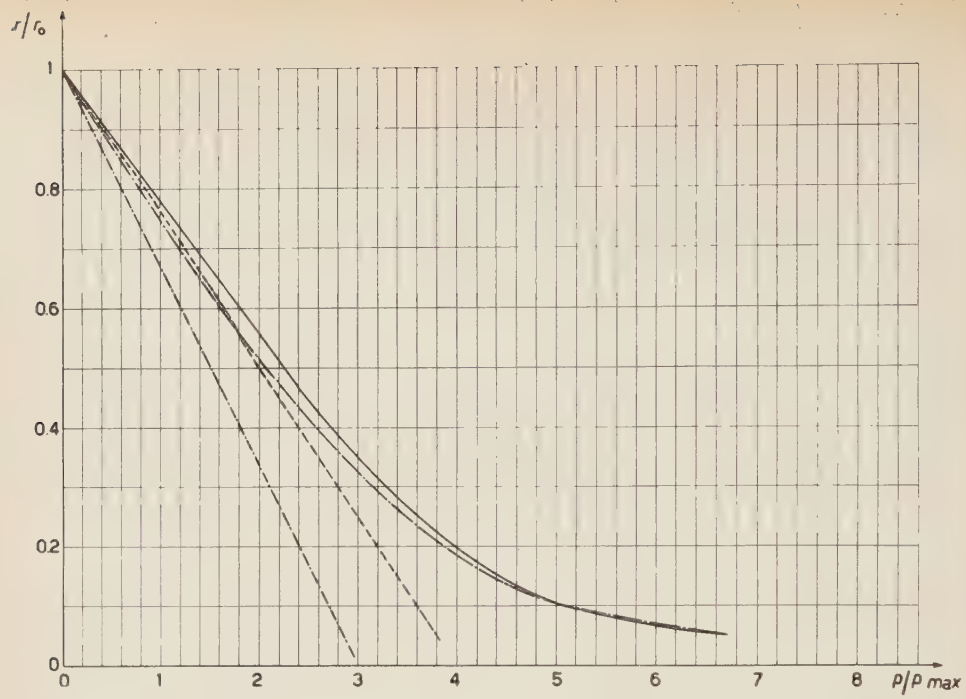

Fig. 10*a*. – Comparison of the exact and approximate equations for meteoric radius as function of the pressure. First approximation: –·–·–; Weiss approximation: – – –. Exact solution: ——. New approximation: —·—·—. $v_0 = 20$ km/s.

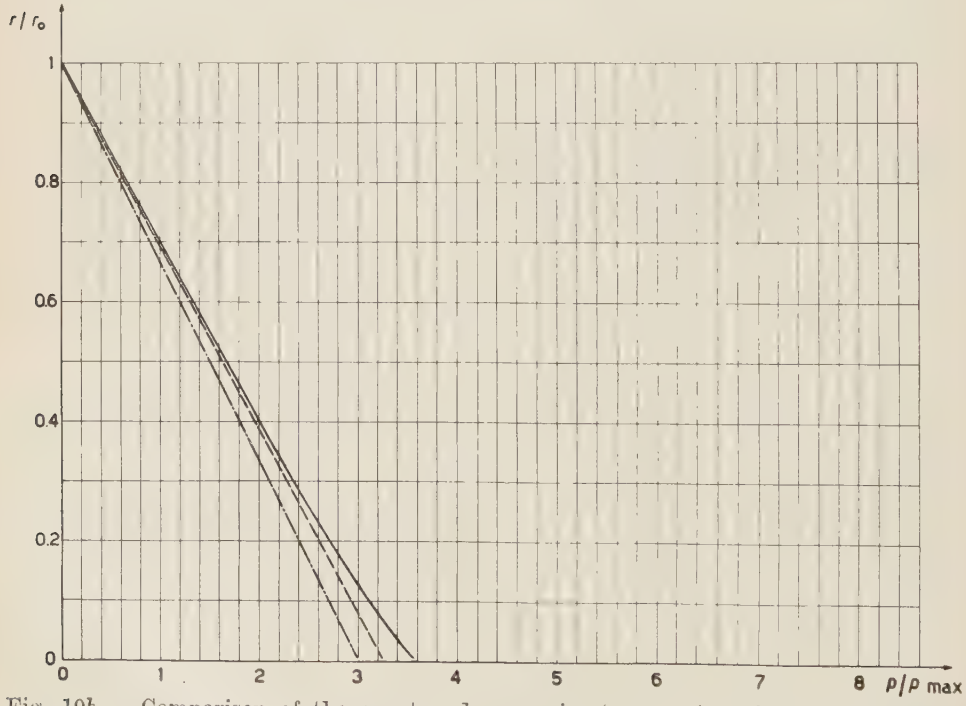

Fig. 10*b*. – Comparison of the exact and approximate equations for meteoric radius as function of the pressure. First approximation: –·–·–; Weiss' approximation: – – –. Exact solution: ——. $v_0 = 40$ km/s.

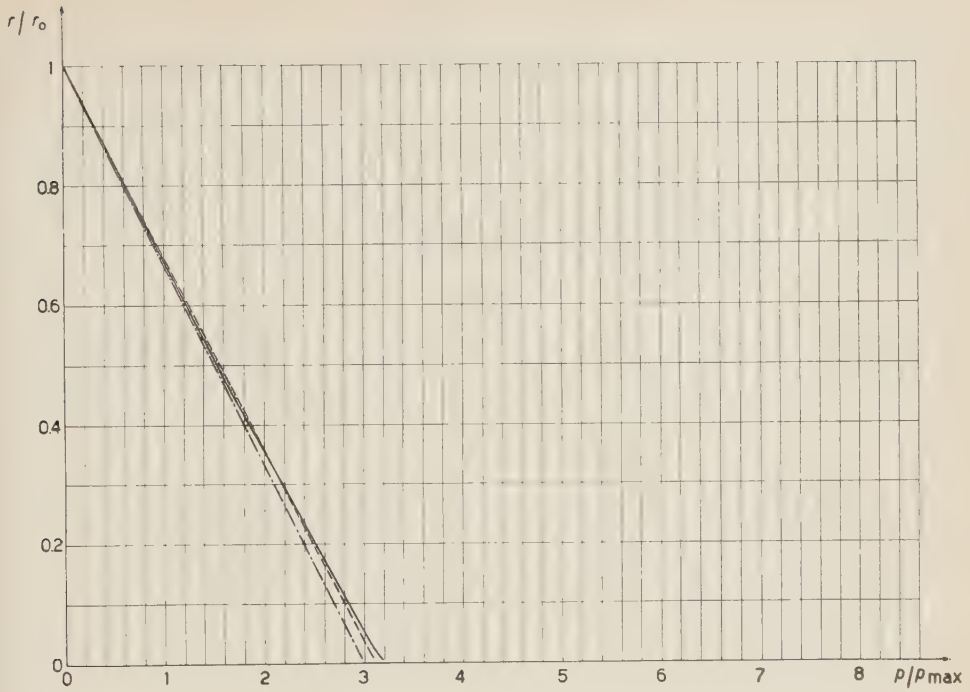

Fig. 10*c*. – Comparison of the exact and approximate equations for meteoric radius as function of the pressure. First approximation: –·–·–; Weiss' approximation: – – –. Exact solution: ———. $v_0 = 60$ km/s.

Consequently, for very slow meteors, we reach the following expressions:

$$q = q_{\max} \frac{\psi}{\psi_{\max}} \left[\frac{1-\psi}{1-\psi_{\max}}\right]^2 \left[\frac{x_0 + \ln(1-\psi)}{x_0 + \ln(1-\psi_{\max})}\right]^{(\eta+2)/2}, \tag{61}$$

$$I = I_{\max} \frac{\psi}{\psi_{\max}} \left(\frac{1-\psi}{1-\psi_{\max}}\right)^2 \left[\frac{x_0 + \ln(1-\psi)}{x_0 + \ln(1-\psi_{\max})}\right]^3, \tag{62}$$

$$a = a_{\max} \frac{p}{p_{\max}} \frac{1-\psi_{\max}}{1-\psi} \frac{x_0 + \ln(1-\psi)}{x_0 + \ln(1-\psi_{\max})}, \tag{63}$$

$$r = r_{\max} \frac{1-\psi}{1-\psi_{\max}}, \tag{64}$$

where

$$\psi = A \frac{p}{\varrho p_{\max}} - B \left(\frac{p}{\varrho p_{\max}}\right)^2. \tag{65}$$

As shown in the figures, the above approximation is close to the exact solution, even when $p/\varphi p_{max}$ is relatively large and other approximations fail.

It is well known that JACCHIA ([13]) has advanced the phenomenon of fragmentation to explane some anomalies in atmospheric densities derived from photographic measurements on meteors; obviously when fragmentation is present, the preceding theory fails, but, at present is doesn't seem that fragmentation be widespread amongst faint radio meteors ([26,38]).

* * *

This research has been sponsored in part by the Air Research and Development Command, United States Air Force, through the European Office, ARDC (Contract AF 61 (052)-227).

([38]) A. A. WEISS: *Austr. Journ. Phys.*, **13**, 532 (1960).

RIASSUNTO

L'attuale teoria dell'ablazione delle meteore viene esaminata criticamente e, in alcuni punti, migliorata. Vengono date espressioni, che approssimano le equazioni fondamentali dell'ablazione con maggior precisione di quelle usate correntemente, particolarmente utili per meteore a velocità relativamente bassa. Sono proposti nuovi metodi per determinare il calore effettivo di ablazione e la densità delle meteore. I risultati ottenuti usando questi metodi saranno pubblicati prossimamente.

IL NUOVO CIMENTO Vol. XIX, N. 3 1° Febbraio 1961

Causality and the Lorentz Group (*).

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(ricevuto il 18 Luglio 1960)

Summary. — A definition of causality, different from the one currently employed in field theory, is introduced. Its relation to the Lorentz group and to relativity theory is clarified. It leads to a reduction of the S-matrix, appropriate to dispersion theory, that is not subject to some limitations hitherto encountered. The mathematical constructs appearing in the course of this reduction are discussed.

1. – Introduction.

Among the more promising recent attempts in field theory are those that exploit the principle of causality. The matrix element for elastic scattering is represented as a Fourier transform of the expectation value of a retarded commutator. The latter is then required to vanish for fields with spacelike separation. This, together with the assumption of a complete set of positive-frequency timelike eigenstates for the displacement operator, leads to interesting analytic properties of the scattering amplitude [1]. In order to deal with processes of greater complexity the single-commutator representation is generalized to one with multiple commutators [2]. This extension is subject to two limitations: there must be no more than two particles in one of the states, and the one-particle states are taken to be stable.

In this article we formulate the principle of causality in a fashion that appears to us more apposite to the context in which it is used. Basing

(*) Work done under the auspices of the U. S. Atomic Energy Commission.

(1) H. Bremerman, R. Oehme and J. Taylor: *Phys. Rev.*, **109**, 2178 (1958).

(2) H. Lehmann, K. Symanzik and W. Zimmerman: *Nuovo Cimento*, **6**, 319 (1957) (LSZ).

ourselves on this definition, we proceed to discuss the relation between causality and the Lorentz group. The discussion suggests a reduction of the S matrix essentially different from the multiple-commutator representations. It is not subject to limitations on the number or stability of particles entering into or emerging from a reaction. Casuality so formulated is then shown to imply microcausality, a condition on a bilinear form in the Bethe-Selpeter amplitudes appropriate to the initial and final states.

2. – Causality in classical particle mechanics.

Let a simultaneous configuration of a system of n particles, subject to a dynamics D, be specified by $z_{(s)}{}^{\mu}(\mu = 0, 1, 2, 3;\ g_{\mu\nu} = 1, -1, -1, -1;\ s = 1, 2, \ldots n)$. Consider two distinct simultaneous configurations $z(\tau_1)$, $z(\tau_2)$ which may evolve from each other according to D. Let the D be thought of as including the mutual interactions of the particles and the effects of external fields. We associate with it a D_0 which is thought of as excluding these and representing the free dynamics of the system. We call the external fields and the mutual interactions causal if $z(\tau_1)$ and $z(\tau_2)$ are also accessible from each other by the motion D_0. Respecting the nature of the latter, this definition has nothing to say.

Applied to a single particle, whose D_0 is a forward motion in time controlled by Newton's equations, the definition would rule as acausal those external fields that result in a later position outside the forward light cone of the original. For a D_0 that admits a forward as well as a backward motion in τ, the class of causal external fields might be broader.

It is difficult to harmonize this denotation of causality with the connotations and associations the word has collected. A more felicitous term, reflecting our anxiety lest certain facets of free motion — to which we are addicted — be effaced by interations, would be desiderable. The more so, since the range of applicability of the word is now limited to theories with a prepossession for sundering interaction effects from the entire motion.

Does the word, with its common denotative value, have any meaning in physics? For a closed dynamical system, unaffected by external forces and without hidden degrees of freedom, we believe not. The central problem for these systems is that of determinism, rather than causality. We are given the law of evolution of the system in the small and inquire what additional conditions (initial, final, boundary) must be imposed for the determination in the large. More precisely the question — purely mathematical — is whether with a certain type of conditions the problem is well set (in the sense of Hadamard) and not whether these refer to the past or future. On the other hand, in considering systems subject to external agencies we are averse to

basing our solution on promises, and account only the past performance of the agents. Since latter can be comprehended in a larger closed system and codetermined with the system under consideration, the concept may well be only ancillary to classical physics. Quantum theory may present a somewhat different picture, since an observer cannot be co-determined with the system he observes and still have any knowledge of it.

Historically, the theory of relativity was the carrier of an idea, somewhat less general and explicit, that comes under the head of our definition of causality. We shall give now a crude operational picture of Lorentz invariance, unembarrassed by it. Within its frame we shall also fit the definitions and discussions of the following sections.

A physicist has somehow come by an album of sketches of orbits of particles. He may have obtained it from observation, by integrating equations of motion, or drawn them from his imagination. To a rather exclusive exhibit of these he invites that public which sees eye to eye with him on certain things. To qualify for an invitation, each one must share his opinions on how to calculate lenghts of vectors from their components and agree with him on other things to be discussed in the next section. He exposes one of his sketches (say a circle) in their view. Each of them reports back his distorted impression of this object (to each of them it may appear as a different ellipse). If he can match each of their distortions with some sketch of his album (« I *have* one like it, but it isn't the one I'm now showing ») and if he can do this for every sketch of his album, then his collection is invariant under the distortion of this group of viewers. In this formulation the invariance of the collection is contingent on the wealth of possibilities of motion the album contains within it rather than on any dynamical details. This picture is made more precise in the next section.

3. – Causality in quantum mechanics.

The homologues of free and interacting classical motions between two configurations are the free $(x_1 \dots x_m | s | y_1 \dots y_n)$ and interacting $(x_1 \dots x_m | S | y_1 \dots y_n)$ transition amplitudes. They are not expected to exhibit the specific features of point mechanics. Even for a single free field the transition amplitude

$$(1) \qquad (x | s | y) = (\Omega, A^-(x) A^+(y) \Omega) = \Delta^+(xy)$$

does not vanish outside the light cone. A general feature of the classical situation, closely relating to the Lorentz group, is, however, reflected in quantum theory. It is that a free particle can be saddled with a Lorentz observer. Our causality requirement restricted the admissible external fields

open for the particle to those in which it still may be tamed. We express this in quantum language by saying that the free-transition amplitude $(x|s|y)$ admits the Lorentz group and we restrict the admissible interactions to those in which $(x|S|y)$ also admits it. In terms of the operational statement of the previous section: all those who agree among themselves on how to compute lengths of vectors do not distort s. To preserve this unanimity, we refrain from introducing controversial interactions that would lead to a distortion of S.

We now proceed to give mathematical form to the idea of distortion as applied to objects of field theory, to list the classes of objects on which the Lorentz public must agree by the very nature of relativity theory, and to show that S is not included among these classes.

The representation of a distortion is well known in the theory of groups of transformations ([3]). Let g be some element of such a group. Let $F(x)$ be a function of the coordinates x whose numerical value is specified in a certain set of frames of reference. It may be a tensor, spinor, Hilbert space vector, or some combination of these. The F as distorted by g, F_g, may be expressed symbolically in terms of the undistorted F through the relation $F_g = gFg^{-1}$, understood to mean that we take g on the left of F in those representations that are appropriate to the tensor or spinor character of F, and g^{-1} on the right in those appropriate to the realization of g on the coordinate variable. For a one-parameter (t) subgroup we denote the latter by x_t and its inverse by x_{-t}. The infinitesimal representation associated with tensor indices is then the matrix $J_t{}^{\mu}{}_{\nu}(x) = \partial x_t{}^{\mu}/\partial x^{\nu}$; the representations s_t associated with spin and U_t with Hilbert space indices are taken, for the moment, as independent of J. The distortion of a c-function tensor is then

$$g^{\mu\nu}(x) \to g_t{}^{\mu\nu}(x) \equiv J_t{}^{\mu}{}_{\mu'} J_t{}^{\nu}{}_{\nu'} g^{\mu'\nu'}(x_{-t}) , \tag{2}$$

a spinor c function is

$$\psi(x) \to \psi_t(x) \equiv s_t \psi(x_{-t}) , \tag{3}$$

a spin operator is

$$\gamma^{\mu}(x) \to \gamma_t{}^{\mu}(x) \equiv J_t{}^{\mu}{}_{\nu} s_t \gamma^{\nu}(x_{-t}) s_t{}^{-1} , \tag{4}$$

a Hilbert space vector is

$$\Omega(x) \to \Omega_t(x) = U_t \Omega(x_{-t}) , \tag{5}$$

([3]) J. A. Schouten: *The Ricci-Calculus* (Berlin, 1954).

a scalar field variable (Hilbert space operator) is

$$\varphi(x) \to \varphi_t(x) = U_t \varphi(x_{-t})\, U_t^{-1}\,. \tag{6}$$

The quantities s and U, in general independent of J and the finite transformation on x, are in field theory related to these by restricting the class of admissible transformations to those which do not distort certain objects.

We shall list the types of objects that are required to be undistortible by Lorentz invariance and show that the transition amplitude is in general not included among them. The primary object of this character is the metric tensor. Its undistortibility defines the representation of the Lorentz group on tensor indices through the requirement $g_t^{\mu\nu}(x) = g^{\mu\nu}(x)$. From now on we confine ourselves to Minkowski frames in which g is diagonal and independent of x. To correlate s to J we also require that the spin operators γ^μ remain undistorted, thus defining the spin representations of the group. The Hilbert space representation U_t is defined by the requirement that all field variables $\varphi(x)$ remain undistorted. Since these are the three types of representation we need in field theory, Lorentz invariance alone does not compel us to postulate any further undistortible objects. In this sense causality, although intimately connected with the mappings under the Lorentz group, is quite independent of the requirements of relativity theory.

Let us now consider the state vector

$$\Omega(x,\,\sigma) = \int_\sigma \mathrm{d}\sigma^\mu(\xi)\, \Delta^+(x-\xi)\, \delta_\mu(\xi)\, \varphi(\xi)\Omega\,, \tag{7}$$

where Ω is the vacuum state, $f\delta_\mu\, g = f\partial_\mu g - (\partial_\mu f)g$, and φ a scalar field operator. For a free field this integral is independent of σ and may be evaluated with x on σ to yield $\omega(x) = \varphi^+(x)\,\Omega$. We then have, by (5),

$$\omega_t(x) = U_t\, \omega(x_{-t}) = U_t \varphi^+(x_{-t})\, U_t^{-1}\, \Omega = \varphi_t^+(x)\, \Omega = \omega(x)\,, \tag{8}$$

from the requirement that φ be not distorted. Relation (8) does not hold for a general $\Omega(x,\,\sigma)$. Geometric considerations indicate that $\Omega_t(x\sigma)$, regarded as a functional of the pseudonormal to the surface σ, is different from $\Omega(x,\,\sigma)$ Since the transition amplitude is a scalar product of state vectors of this type with $\sigma \to \pm\infty$, it need not admit the group for transitions involving interaction.

We shall accordingly reduce S by expanding it in terms of s. The exacting demand of admitting the group is then passed on to the coefficients of the expansion. These turn out to be bilinear forms in the BS amplitudes. To carry out this project, we need three lemmata. The next three sections are

devoted to their derivation. The first lemma is also of some intrinsic interest. It permits us to compare the reduction scheme of dispersion theory with those employed in other approaches to field theory. The second is a device for generating elements of s. The third relates to generalized retarded commutators and implies a certain reciprocity relation for the internal orderings of factors in a product of fields. These in turn suggest a view on dispersion theory, discussed briefly in the final section.

4. – Dirac identities.

The object of the work in this section is to derive the formula on which our reduction of the S matrix is based. It is an extension of Dirac's $A^{\text{out}}=A^{\text{in}}+A^{\text{rad}}$ to second quantized systems. In the extended version A^{out} and A^{in} are products of outgoing and incoming operators as they occur in the construction of the outgoing and incoming state vectors. Two sets of quantities corresponding to A^{rad} are defined and have in common with it an essential property. A notation requiring somewhat elaborate definitions is introduced to make the final expression compact ad explicit.

We consider a specific matrix element of the scattering operator with $m \geqslant 1$ incoming and $n \geqslant 1$ outgoing, not necessarily distinct, scalar fields. Let these be arbitrarily ordered as in

$$\Omega_i \cdot (x', x'', x''', \ldots) = \psi_i^+(x') \varphi_i^-(x'') \chi_i^+(x''') \chi_i^-(x^{\text{iv}}) \ldots \Omega ,$$

$$\Omega_o \cdot (y', y'', y''', \ldots) = \Sigma_o^+(y') A_o^+(y'') \Theta_o^+(y''') A_o^+(y^{\text{iv}}) \ldots, \Omega ,$$

where the superscript $(+)$ denotes the positive frequency part of the incoming (i) or outgoing (o) field and the vacuum state is assumed to be stable. The ordered sequence with repetition in the field variables is now placed in a one-to-one correspondence with sequences of field variables without repetition. These are $A_1^+(x_1)$, $A_2^+(x_2)$, ... $A_m^+(x_m)$ and $B_1^+(y_1)$, $B_2^+(y_2)$, ... $B_n^+(y_n)$ for the field operators of Ω_i and Ω_0 respectively. A permutation group $\boldsymbol{p}$ on m letters is now associated with the A sequence, and $\boldsymbol{q}$ on n letters with B. The domains of the permutations are the identical subscripts of the field operator and coordinate variable. A particular permutation p may be defined by

$$(9) \quad p[A_1(x_1), A_2(x_2), A_3(x_3), \ldots A_m(x_m)] = [A_3(x_3), A_1(x_1), A_m(x_m), \ldots, A_2(x_2)] .$$

The definition of a multifield operator constructed on a sequence of *com-*

muting operators (like A_i^+) is given by the formulas

$$(10)\quad \begin{cases} {}^{sp1}A(x) = 1 & \text{for } s = 0, \\ {}^{sp1}A(x) = {}^{p1}A(x_1, x_2, \ldots x_s) = A_{p1}(x_{p1})\, A_{p2}(x_{p2}) \ldots A_{ps}(x_{ps}) & \text{for } 1 \leqslant s \leqslant m. \end{cases}$$

The first of these is a convention. In the second, the first equality sign translates a compact into a more prolix notation; the second equality explicates these in terms of elementary field variables. In words: to construct ${}^{sp1}A(x)$ permute the standard sequence by p and form the product of the first s elements of the permuted sequence. It will useful to have a notation ${}^{sp1}A'(x)$ for the complement of ${}^{sp1}A(x)$:

$$(10')\qquad {}^{sp1}A'(x) = {}^{p1}A(x_{s+1}, x_{s+2}, \ldots x_m) = A(x_{p(s+1)}, x_{p(s+2)}, \ldots x_{pm})\,.$$

The identity permutation is not indicated; thus ${}^{s1}A(x) = A_1(x_1) \ldots A_s(x_s)$. We also omit the s for $s=1$; thus ${}^{p1}A(x) = A_{p1}(x_{p1})$. Consistently, then, $A(x) = A_1(x_1)$. We note ${}^{mp1}A(x) = {}^{m}A(x)$ for all p. By way of example, with p of (9) we have

$$\begin{aligned} {}^{p1}A(x) &= A_3(x_3)\,, \\ {}^{2p1}A(x) &= A_3(x_3)\, A_1(x_1)\,, \\ {}^{3p1}A(x) &= A_3(x_3)\, A_1(x_1)\, A_m(x_m)\,. \end{aligned}$$

In order to extend these definitions to sequences of non-commuting operators we introduce a system of θ functions.

$$(11)\quad \begin{cases} {}^{sp1}\vartheta(x) = 1 & \text{for } s = 0\,, \\ \qquad = 1 & \text{for } s = 1\,, \\ \qquad = \theta(x_{p1}\,|\,x_{p2}\,|\ldots|\,x_{ps}\,|) = {}^{p}\vartheta(x_1\,|\,x_2\,|\ldots x_s) & \text{for } s > 1\,, \end{cases}$$

where

$$\begin{aligned} {}^{p}\theta_+(x_1\,|\ldots|\,x_s) &= 1 && \text{for } x^0_{p1} < \ldots < x^0_{ps}\,, \\ &= 0 \text{ otherwise}, \\ {}^{p}\theta_-(x_1\,|\ldots|\,x_s) &= 1 && \text{for } x^0_{p1} > \ldots > x^0_{ps}\,, \\ &= 0 \text{ otherwise}. \end{aligned}$$

These are direct generalizations of ${}^{2}\theta(x) = \theta(x_1 - x_2)$, and may be expressed in terms of this symbol; thus ${}^{3}\theta(x) = \theta(x_1 - x_2)\,\theta(x_2 - x_3)$. We associate with this set a permutation group $\boldsymbol{\pi}$ on s letters whose domain consists of the

subscripts of x in θ:

$$ {}^{\pi sp1}\theta(x) = \theta(x_{\pi p1}|x_{\pi p2}|\dots|x_{\pi ps}) . $$

From the meaning of these symbols it is readily seen that

$$ \sum_{\pi} {}^{\pi sp1}\theta_{\pm}(x) = 1 . \tag{12} $$

We now define the multifield operator ${}^{sp}A_{\pm}(x)$ on the ordered set of simple operators $[A_1(x_1), A_2(x_2), \dots A_m(x_m)]$ by the formula

$$ \begin{cases} {}^{sp}A_{\pm}(x) = 1 & \text{for } s = 0 , \\ {}^{sp}A_{\pm}(x) = \sum_{\pi} {}^{\pi sp1}\theta_{\pm}(x)\ {}^{\pi sp1}A(x) & \text{for } m \geqslant s \geqslant 1. \end{cases} \tag{13} $$

In words: to construct ${}^{sp}A$ apply a p permutation to the sequence, form the product ${}^{sp1}A$ of the first s elements from left to right, multiply by an appropriate θ, and sum over all possible temporal arrangements. We note in particular the relation ${}^{1}A_{+} = {}^{1}A_{-}$. If the operators in the sequence commute we have ${}^{\pi sp1}A(x) = {}^{sp1}A(x)$ and, by (12), ${}^{sp}A(x) = {}^{sp1}A(x)$. Definition (13) then includes (10) as a special case. It is the p bracket of Dyson, the $\pm$ brackets of Schwinger, the T product of Wick, the T function of LSZ [3], etc., defined on the sp subset of the set of m operators. We enriched this collection with one more notation, which is handled more smoothly in algebraic manipulations.

The Dirac definition of the radiation field $\mathscr{A}(x)$ given by the expression

$$ \underset{o}{A}(x) = \underset{i}{A}(x) + \mathscr{A}(x) \tag{14} $$

may be regarded as a formal identity:

$$ \underset{o}{A}(x) = -\int_{-\infty} \mathrm{d}\sigma_{\mu}(\xi)\ \vartheta(x-\xi)\,\delta_{\mu}(\xi)\,A(\xi) = -\left(\int_{-\infty} - \int_{\infty}\right) \mathrm{d}\sigma^{\mu}(\xi)\ \vartheta(x-\xi)\,\delta_{\mu}(\xi)\,A(\xi) + $$

$$ + \int_{-\infty} \mathrm{d}\sigma_{\mu}(\xi)\,\Delta(x-\xi)\,\delta_{\mu}(\xi)A(\xi) = \underset{i}{A}(x) + \int \mathrm{d}\sigma(\xi)\,\Delta(x-\xi)\,K(\xi)\,A(\xi) = \underset{i}{A}(x) + \mathscr{A}(x) . $$

In this expression $\Delta(x)$ is the radiation kernel appropriate to the A field and K the Gordon-Klein operator that annihilates it. To have a more compact notation we write

$$ \int_{\sigma} \mathrm{d}\sigma^{\mu}(\xi)\ \vartheta(x-\xi)\,\delta_{\mu}(\xi) \dots = -\int_{\sigma} \mathrm{d}H(x-\xi) \dots , $$

$$ \int \mathrm{d}\sigma(\xi)\ \vartheta(x-\xi)\,K(\xi) \dots = -\int \mathrm{d}K(x-\xi) \dots , $$

where H is, essentially, the Huygens kernel and K relates the radiation field to its source. In this notation, an abbreviated form of that in LSZ, Green's theorem for a c function $f(x)$ is

$$(15_0) \qquad \int_{\sigma_1}^{\sigma_2} \mathrm{d}K(x-\xi)\, f(\xi) = \left(\int_{\sigma_2} - \int_{\sigma_1}\right) \mathrm{d}H(x-\xi)\, f(\xi)$$

if $f(x)$ vanishes sufficiently rapidly on the timelike remote surfaces. We adapt now one of their principal techniques to extend (15_0) to cases in which f is replaced by the ordered operators ${}^sA_\pm$ of (13). Clearly (15_0) still holds for ${}^sA$ with $s=1$. For ${}^sA$ with $s>1$ we need the following versions of Green's theorem:

$$(15_+) \qquad \int \mathrm{d}K(x_n-\xi_n)\, A_+(x_1 \dots x_{n-1}\xi_n) = A_+(x_1 \dots x_{n-1})\, \underset{o}{A}(x_n) - \underset{i}{A}(x_n) A_+(x_1 \dots x_{n-1}) ,$$

$$(15_-) \qquad \int \mathrm{d}K(x_n-\xi_n)\, A_-(x_1 \dots x_{n-1}\xi_n) = \underset{o}{A}(x_n) A_-(x_1 \dots x_{n-1}) - A_-(x_1 \dots x_{n-1})\, \underset{i}{A}(x_n) .$$

By the argument leading to (15_0) we have

$$\int \mathrm{d}K(x_n-\xi_n)\, A_+(x_1 \dots x_{n-1}\xi_n) = \left(\int_{\xi_n^0=+\infty} - \int_{\xi_n^0=-\infty}\right) A_+(x_1 \dots x_{n-1}\xi_n)\, \mathrm{d}H(x_n-\xi_n) ,$$

and according to (13),

$$A_+(x_1 \dots x_{n-1}x_n) = \sum_\pi \theta_+(x_{\pi 1}|\dots|x_{\pi(n-1)}|x_{\pi n})^{\pi 1}A(x) \dots {}^{\pi(n-1)}A(x)^{\pi n}A(x) .$$

We see then that only permutations with fixed $\pi(n)=n$ and $\pi(1)=n$ contribute to the upper and lower limits respectively. Equation (15_+) then follows from the definitions of $\underset{o}{A}$ and $\underset{i}{A}$. Similar considerations lead to (15_-).

The definitions

$$(16a1) \qquad \underset{o(i)}{A}(x) = \int_{-\infty(-\infty)} \mathrm{d}H(x-\xi)\, A(\xi) ,$$

$$(16b1) \qquad \mathscr{A}_\pm(x) = \int \mathrm{d}K(x-\xi)\, A_\pm(\xi)$$

of the quantities entering into Dirac's expression are now regarded as par-

ticular cases of the more general

$$(16a) \qquad {}^{sp}_{o(i)}A(x) = \int_{+\infty(-\infty)} {}^{sp}\mathrm{d}H(x-\xi)\,{}^{sp}A(\xi)$$

$$(16b) \qquad {}^{sp}\mathscr{A}_{\pm}(x) = \int {}^{sp}\mathrm{d}K(x-\xi)\,{}^{sp}A_{\pm}(\xi)\,.$$

We take the usual asymptotic conditions to hold and therefore do not provide the A on the right-hand side of (16a) with an ordering $(\pm)$.

Definition (16a) is a direct consequence of (16a1). The designation of (16b) as a radiation operator is varranted on the ground that, as also in (16b1), its positive and negative frequency projections vanish in the vacuum state. These projections ${}^{s}_{o(i)}A^{\pm}$, ${}^{s}\mathscr{A}^{\pm}_{\pm}$ are obtained through the integral transforms (16) with kernels $H^{\pm}$, $K^{\pm}$ in which all the Δ functions are replaced by Δ^{+} or all the Δ functions by Δ^{-}. Since Δ is $\Delta^{+}+\Delta^{-}$, we have

$$ {}^{s}\mathrm{d}H = {}^{s}\mathrm{d}H^{+} + {}^{s}\mathrm{d}H^{-} \qquad \text{for } s=1,$$

$$ {}^{s}\mathrm{d}H \neq {}^{s}\mathrm{d}H^{+} + {}^{s}\mathrm{d}H^{-} \qquad \text{for } s>1,$$

with similar relations for K. The statements

$$(17) \qquad \Omega^{-s}\mathscr{A}^{+}_{+}\Omega^{+}=0, \quad \Omega^{-s}\mathscr{A}^{+}\Omega^{+}=0, \quad \Omega^{-s}\mathscr{A}^{-}_{+}\Omega^{+}=0, \quad \Omega^{-s}\mathscr{A}^{-}\Omega^{+}=0$$

obviously hold for $s=1$ (with the assumed stable vacuum). The $(-)+$ on Ω in (17) designates a (∞) contravariant vector and thus obviates the necessity of brackets in denoting scalar products. We shall adhere to this convention. Applying K integral transforms to the variables $x_1 \dots x_{n-1}$ in (15), we obtain

$$(18_{+}) \qquad {}^{s}\mathscr{A}_{+}(x) = {}^{s-1}\mathscr{A}_{+}(x)\,\underset{o}{A}(x_s) - \underset{i}{A}(x_s)\,{}^{s-1}\mathscr{A}_{+}(x)\,,$$

$$(18_{-}) \qquad {}^{s}\mathscr{A}_{-}(x) = \underset{o}{A}(x_s)\,{}^{s-1}\mathscr{A}_{-}(x) - {}^{s-1}\mathscr{A}_{-}(x)\,\underset{i}{A}(x_s)\,.$$

The positive-frequency projection of (18_{+}), recursively expandend and simplified by $\Omega^{-+}\underset{i}{A}=0$, yields

$$\Omega^{-s}\mathscr{A}^{+}_{+}(x)\,\Omega^{+} = \Omega^{-(s-1)}\mathscr{A}^{+}_{+}(x)\,\underset{o}{A}{}^{+}(x_s)\,\Omega^{+} = \Omega^{-(s-2)}\mathscr{A}^{+}_{+}(x)\,\underset{o}{A}{}^{+}(x_{s-1}x_s)\Omega^{+} =$$

$$= \dots = \Omega^{-}\mathscr{A}^{+}(x_1)\,\underset{o}{A}{}^{+}(x_2 \dots x_n)\,\Omega^{+}=0\,.$$

The remaining three assertions of (17) may be deduced in a similar fashion.

The two pairs of expansion formulas for ${}^{m}_{0}A$ and ${}^{m}_{i}A$ are written down as

$$(D_1a) \qquad {}^{m}_{o}A(x) = \frac{1}{m!}\sum_{\sigma}\binom{m}{\sigma}\sum_{p}{}^{\sigma p}_{i}A'(x)\,{}^{\sigma p}\mathscr{A}_{+}(x)\,,$$

$$(D_1b) \qquad {}^{m}A(x) = \frac{1}{m!}\sum_{\sigma}\binom{m}{\sigma}\sum_{p}{}^{\sigma p}_{i}A(x)\,(-1)^{\sigma'\sigma p}\mathscr{A}'_{-}(x)\,;$$

$$(D_2a) \qquad {}^{m}_{o}A(x) = \frac{1}{m!}\sum_{\sigma}\binom{m}{\sigma}\sum_{p}{}^{\sigma p}\mathscr{A}_{-}(x)\,{}^{\sigma p}_{i}A'(x)\,,$$

$$(D_2b) \qquad {}^{m}_{i}A'(x) = \frac{1}{m!}\sum_{\sigma}\binom{m}{\sigma}\sum_{p}(-1)^{\sigma\sigma p}\mathscr{A}_{+}(x)\,{}^{\sigma p'}_{o}A'(x)\,.$$

The grouping into pairs is motivated by the fact that from a more detailed version of D (with expansions for ${}^{sp}A$) it is possible to obtain a completeness relation — a bilinear form involving only $\mathscr{A}_{\pm}$ — for each pair. Let $p(\sigma)$ be the stability subgroup of p that leaves $(1, 2, \ldots \sigma)$ invariant and p/σ the set of equivalence classes of p defined by $p(\sigma)$. Since A and $\mathscr{A}$ are symmetric in their arguments, we may convert the sum over all the elements of the group into one over the equivalence classes by the corresponding stability subgroup. For (D_1a) we then obtain

$$(D'_1a) \qquad {}^{m}_{o}A(x_1 \ldots x_m) = \sum_{\sigma=0}^{m}\sum_{p/\sigma}{}^{p}_{i}A(x_1, \ldots, x_\sigma)\,{}^{p}\mathscr{A}_{+}(x_{\sigma+1}, \ldots x_m)\,.$$

This reduces for $m = 1$ to ${}_{o}A(x_1) = {}_{i}A(x_1) + \mathscr{A}_{+}(x_1)$, the correct identity. Assume that (D'_1a) holds for $m = n-1$ and multiply both sides from the right by $A(x_n)$:

$$(19) \qquad {}^{n}_{o}A(x_1 \ldots x_n) = \sum_{\sigma=0}^{n-1}\sum_{p/\sigma}{}^{p}_{i}A(x_1, \ldots x_\sigma)\,{}^{p}\mathscr{A}_{+}(x_{\sigma+1}, \ldots x_{n-1})\,{}_{o}A(x_n)\,.$$

In (19) p is a permutation on $n-1$ letters. With the aid of (18_{+}):

$$(20) \qquad {}^{n}_{o}A(x_1 \ldots x_n) = \sum_{\sigma=0}^{n-1}\sum_{p/\sigma}[{}^{p}_{i}A(x_1 \ldots x_\sigma)\,{}^{p}\mathscr{A}_{+}(x_{\sigma+1}, \ldots x_{n-1}x_n) + \\ + {}^{p}_{i}A(x_1, \ldots x_\sigma x_n)\,{}^{p}\mathscr{A}_{+}(x_{\sigma+1}, \ldots x_{n-1})]\,.$$

The summation of the first term in the bracket covers the subset of p/σ on n letters that leaves x_n alone; of the second its complement. The two may be combined into a single term with p/σ on n letters and represented in the form (D'_1a) with σ running from zero to n. The remaining three expansions are deduced in a similar fashion.

5. – Free-field commutators.

In this sections we derive expansions for commutators of free multifield operators acting on the vacuum. These expressions identified with either incoming or outgoing fields are used in a later section.

Our starting point is the observation that $[A^-(x),\ B^+(y)]$ may be regarded as an element of the free-particle s matrix $(x|s|y)$, and the familiar expansion

$$[A^-(x),\ B_1^+(y_1)B_2^+(y_2)\ldots B_n^+(y_n)] = [A^-(x),\ B_1^+(y_1)]B_2^+(y_2)\ldots B_n^+(y_n) + \ldots$$

may therefore be rewritten as

$$[A^-(x),\ {}^nB^+(y)] = \sum_{q/1} (x|s|y)^{q\cdot q}B^{+\prime}(y) = \sum_{\sigma=0}^{n}\sum_{q/\sigma} (x|s|y)^{q\sigma\ \sigma q}B^{+\prime}(y)\ . \tag{21}$$

The second equality is valid because the s operator between states with different numbers of fields vanishes. This, and a corresponding relation, may therefore be expressed as

$$(E_+^0) \qquad [A^-(x),\ {}^nB^+(y)] = \frac{1}{n!}\sum_{\sigma}(\overset{n}{\sigma})\sum_{q}(x|s|y)^{q\sigma\ \sigma q}B^{+\prime}(y)\ ,$$

$$(E_-^0) \qquad [{}^mA^-(x),\ B^+(y)] = \frac{1}{m!}\sum_{\sigma}(\overset{m}{\sigma})\sum_{p}{}^{\sigma p}A^{-\prime}(x)\,{}^{\sigma p\prime}(x|s|y)\ .$$

Contracting E_+^0 with $\omega_-(x_2\ldots x_n)$, ω_+ and E_-^0 with ω_-, $\omega_+(y_2\ldots y_m)$, we get the recurrences

$$(22a) \qquad {}^n(x|s|y)^n = \frac{1}{n!}\sum_{\sigma}(\overset{n}{\sigma})\sum_{q}(x|s|y)^{q\sigma\,\prime}(x|s|)^{\prime q\sigma}\ ,$$

$$(22b) \qquad {}^n(x|s|y)^n = \frac{1}{n!}\sum_{\sigma}(\overset{n}{\sigma})\sum_{p}{}^{\sigma p\prime}(x|s|y)^{\prime\,\sigma p}(x|s|y)\ ,$$

or — without redundant summations —

$$(22a') \qquad {}^n(x|s|y)^n = \sum_{q/1}(x|s|y)^{q\ \prime}(x|s|y)^{\prime q}$$

$$(22b') \qquad {}^n(x|s|y)^n = \sum_{q/1}{}^{p\prime}(x|s|y)^{\prime\ p}(x|s|y)\ .$$

The contraction of E^0_+ with ω_+ and E^0_- with ω_- gives

$$(E_+) \qquad [{}^m A^-(x), {}^n B^+(y)]\omega_+ = \frac{1}{n!}\sum_\sigma \binom{n}{\sigma} \sum_q {}^m(x\,|\,s\,|\,y)^{q\sigma}\omega_+{}^{\prime\sigma q}(y) - \delta^{m0}\omega^n_+(y),$$

$$(E_-) \qquad \omega_-[{}^m A^-(x), {}^n B^+(y)] = \frac{1}{m!}\sum_\sigma \binom{m}{\sigma} \sum_p \omega'_-{}^{p\sigma}(x)\,{}^{\sigma p}(x\,|\,s\,|\,y)^n - \delta^{n0}\omega^m_-(x)\,,$$

where $m = 1$ in E_+ and $n = 1$ in E_-. We shall now show that E is valid without this restriction. To insure its validity for $m = 0$, $n = 0$ we inserted the terms with the Kronecker deltas.

In the alternative version, E_+ states

$$(23\alpha) \qquad [{}^m A^-(x), {}^n B^+(y)]\omega_+ = \sum_{\sigma=0}^{n} \sum_{q/\sigma} {}^m(x\,|\,s\,|\,y)^{q\sigma}\omega'_+{}^{\sigma q}(y) = 0 \qquad \text{for } m > n,$$

$$(23\beta) \qquad = \sum_{q/m} {}^m(x\,|\,s\,|\,y)^{qm}\omega'_+{}^{mq}(y) \qquad \text{for } 0 < m \leqslant n,$$

where we made use of the orthogonality of initial and final states with different numbers of fields. Assertion (23α) is obviously correct. This is evident from the fact that, because of the presence of ω_+, we can construct a multiple commutator of ${}^n B^+(y)$ with single $A^-(x)$, of which there are enough to devour it. To investigate (23β) we introduce $\nu \geqslant 0$, $n = m + \nu$ and rewrite the expression as

$$(23\beta') \qquad [{}^m A^-(x), {}^{(m+\nu)} B^+(y)]\omega_+ = \sum_{q/m} {}^m(x\,|\,s\,|\,y)^{qm}\omega'_+{}^{mq}(y)\,,$$

valid for $m = 1$ by (21). We assume it to hold for $m = n$ and use recurrence $(22\alpha')$ to prove its validity for $m = n + 1$. In extenso, this is

$$(22\alpha'') \quad (x_1, \ldots, x_{n+1}\,|\,s\,|\,y_1, \ldots, y_{(n+1)}) = \sum_{\pi/1} (x_1\,|\,s\,|\,y_{\pi 1})(x_2, \ldots, x_{n+1}\,|\,s\,|\,y_{\pi 2}, \ldots, y_{\pi(n+1)}) =$$

$$= \sum_{\pi/n} (x_1, \ldots, x_n\,|\,s\,|\,y_{\pi 1}, \ldots, y_{\pi n})(x_{n+1}\,|\,s\,|\,y_{\pi(n+1)})$$

by an obvious change in variables, or

$$(24) \quad (x_1, \ldots, x_{n+1}\,|\,s\,|\,y_{q1}, \ldots, y_{q(n+1)}) = \sum_{\pi/n} (x_1, \ldots, x_n\,|\,s\,|\,y_{\pi q 1}, \ldots, y_{\pi q n})(x_{n+1}\,|\,s\,|\,y_{\pi q(n+1)})\,,$$

where q is any permutation appearing in $(23\beta')$. Substituting this into the right-hand member of the latter for $m = n + 1$, we show, by a sequence of

steps, that it is equal to the left:

$$\sum_{q/(n+1)} {}^{(n+1)}(x\ ,s\,|\,y)^{q(n+1)}\,\omega_+^{\prime q(n+1)}(y) =$$

$$= \sum_{q/(n+1)} (x_1, \ldots, x_{n+1}\,|\,s\,|\,y_{q1}, \ldots, y_{q(n+1)})\,\omega_+(y_{q(n+2)}, \ldots, y_{q(n+\nu+1)}) =$$

$$= \sum_{q/(n+1)} \sum_{\pi/n} (x_1, \ldots, x_n\,|\,s\,|\,y_{\pi q1}, \ldots, y_{\pi qn})\,(x_{n+1}\,|\,s\,|\,y_{\pi q(n+1)})\,\omega_+(y_{q(n+2)}, \ldots, y_{q(n+\nu+1)}) =$$

$$= \sum_{\pi/n} \sum_{q/\pi(n+1)'} (x_1, \ldots, x_n\,|\,s\,|\,y_{\pi 1}, \ldots, y_{\pi n})\,(x_{n+1}\,|\,s\,|\,y_{q\pi(n+1)})\,\omega_+(y_{q\pi(n+2)}, \ldots, y_{q\pi(n+\nu+1)}) =$$

$$= \sum_{\pi/n} (x_1, \ldots, x_n\,|\,s\,|\,y_{\pi 1}, \ldots, y_{\pi n}) \sum_{q/\pi(n+1)'} (x_{n+1}\ s\,|\,y_{q\pi(n+1)})\,\omega_+(y_{q\pi(n+2)}, \ldots, y_{q\pi(n+\nu+1)}) -$$

$$= A^-_{n+1}(x_{n+1}) \sum_{\pi/n} (x_1, \ldots, x_n\,|\,s\,|\,y_{\pi 1}, \ldots, y_{\pi n})\,\omega_+(y_{\pi(n+1)}, \ldots, y_{\pi(n+1+\nu)}) =$$

$$= A^-_{n+1}(x_{n+1})[{}^nA^-(x),\, B^{n+1+\nu}(y)]\,\omega_+ = [{}^{n+1}A^-(x),\, B^{n+1+\nu}(y)]\,\omega_+ .$$

We go from the first member to the second by simply expanding the notation; from the second to the third, by substituting (24). A simple consideration shows the validity of the interchange of the order of summations involved in passing from the third to the fourth member. We can see that the q summation is carried out as indicated in the fifth by noting its equivalence to the action of a destruction operator on a state vector, which leads to the sixth. In passing to the seventh we use the hypothesis that the statement holds for $m = n$. The transition to the eighth involves an elementary manipulation with operators. The same may be done for E_-.

6. – Retarded commutators and reciprocities.

Our goal is to express an arbitrary element of the scattering operator by single retarded commutators of multiple fields. One of the factors of the commutator is constructed on Heisenberg operators of the initial, and the other, of the final state. The Dirac identities provide the link between the incoming and outgoing fields and these factors. In arriving at the Dirac identities we introduced into the expansions (D) for ${}^{in}_{i}A$, ${}^{in}_{o}A$ internal orderings ($\pm$) of the fields pertaining to the initial and final states. These have no physical basis and should not appear in any physically significant context. In this section we derive reciprocity relations which indicate that this is indeed so, and that the only relevant temporal order is that of the initial relative to the final state. To express this order we need another θ symbol,

$${}^m(x\ \theta\,|\,y)^n = (x_1, \ldots, x_m\,|\,\theta\,|\,y_1, \ldots, y_n) = 1, \quad \text{if } x_i^0 \text{ is less than } y_j^0$$

$$\text{for all } 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n,$$

$$= 0, \quad \text{if not,}$$

with the obvious composition law

$$(25)\quad {}^{m}(x\,|\,\theta\,|\,y)^{n}=(x_1\,|\,\theta\,|\,y)^{n}\,(x_2\,|\,\theta\,|\,y)^{n}\ldots(x_m\,|\,\theta\,|\,y)^{n}={}^{m}(x\,|\,\theta\,|\,y_1)\,{}^{m}(x\,|\,\theta\,|\,y_2)\ldots{}^{m}(x\,|\,\theta\,|\,y_n)\,.$$

We start with the identities

$$(26i_{+})\quad [{}^{m}\underset{i}{A}(x),\,{}^{n}F(y)]=(-1)^{\sigma}\int\mathrm{d}K'(x-\xi)^{\sigma'}\,(\xi\,|\,\theta\,|\,y)^{n}\,[{}^{m-\sigma}\underset{i}{A}(x)\,{}^{m-\sigma}A'_{+}(\xi),\,{}^{n}F(y)]\,,$$

$$(26i_{-})\quad [{}^{m}\underset{i}{A}(x),\,{}^{n}F(y)]=(-1)^{\sigma}\int{}^{\sigma}\mathrm{d}K(x-\xi)^{\sigma}\,(\xi\,|\,\theta\,|\,y)^{n}\,[{}^{\sigma}A(\xi)\,{}^{\sigma}\underset{i}{A}'(x),\,{}^{n}F(y)]\,,$$

$$(26o_{+})\quad [{}^{m}F(x),\,{}^{n}\underset{o}{B}(y)]=\int{}^{\sigma}\mathrm{d}K(y-\eta)^{m}\,(x\,|\,\theta\,|\,\eta)^{\sigma}\,[{}^{m}F(x),\,{}^{\sigma}B_{+}(\eta)\,{}^{\sigma}\underset{o}{B}'(y)]\,,$$

$$(26o_{-})\quad [{}^{m}F(x),\,{}^{n}\underset{o}{B}(y)]=\int{}^{n\quad\sigma}\mathrm{d}K'(y-\eta)\,{}^{m}(x\,|\,\theta\,|\,\eta)^{\sigma}\,[{}^{m}F(x),\,{}^{n-\sigma}\underset{o}{B}(y)\,{}^{n-\sigma}B'_{-}(\eta)]\,,$$

where ${}^{n}F(x)=F(x_1,\ldots,x_n)$ is an operator depending on the indicated coordinates. They express the commutator of an incoming or outgoing field with an arbitrary field as an integral transform of a retarded commutator. One of its factors has a part in which the initial or final fields are internally $(\pm)$ ordered. This order depends on the position the fields occupy relative to the unordered part. With the devices of LSZ their derivation is quite simple. We illustrate it by indicating the essential steps that lead to $(26i_{-})$:

$$(27)\quad [{}^{m}\underset{i}{A}(x),\,{}^{n}F(y)]=\int\limits_{-\infty}\mathrm{d}H(x_1-\xi_1)\,[\underset{i}{A}(\xi_1)\,A'(x),\,{}^{n}F(y)]=$$

$$=-\int\mathrm{d}K(x_1-\xi_1)(\xi_1\,|\,\theta\,|\,y)^{n}\,[A(\xi_1)\,\underset{i}{A}'(x),\,{}^{n}F(y)]\,.$$

In going from the second to the third member we ordered ξ_1 relative to the set of y's by means of the θ symbol. With this symbol in, we could with impunity add a surface integral at $+\infty$ and apply Green's theorem. The next step is

$$(28)\quad [A(\xi_1)\,\underset{i}{A}'(x),\,{}^{n}Fy)]=\int\limits_{-\infty}\mathrm{d}H(x_2-\xi_2)\,[A(\xi_1)\,A(\xi_2)\,{}^{2}\underset{i}{A}'(x),\,{}^{n}F(y)]=$$

$$=\int\limits_{-\infty}\mathrm{d}H(x_2-\xi_2)(\xi_2\,|\,\theta\,|\,y)^{n}\,[A_1(\xi_1)\,A_2\,(\xi_2)\,{}^{2}\underset{i}{A}'(x),\,{}^{n}F(y)]-$$

$$-\int\limits_{-\infty}\mathrm{d}H(x_2-\xi_2)(\xi_2\,|\,\theta\,|\,y)^{n}\,[A_2(\xi_2)\,A_1(\xi_1)\,{}^{2}\underset{i}{A}'(x),\,{}^{n}F(y)]=$$

$$=-\int\mathrm{d}K(x_2-\xi_2)(\xi_2\,|\,\theta\,|\,y)^{n}\,[A_{-}(\xi_1\xi_2)\,{}^{2}\underset{i}{A}'(x),\,{}^{n}F(y)]\,,$$

where the third member is obtained from the second by equipping it with a θ factor and substracting a null surface integral at $+\infty$, with the order of the $A(\xi)$ reversed. The fourth member is obtained from the third on the basis of the observation that the order of the $A(\xi)$'s in the surface integrals corresponds to that of $A_{-}(\xi_1\xi_2)$. Substituting (28) into the last member of (27), we notice that the θ factors combine according to (25), and the resulting identity is $(26i_{-})$ with $\sigma = 2$.

We now take $\sigma = m$ in $(26i)$ and $\sigma = n$ in $(26o)$. From this we deduce

$$(29)\quad \int {}^{m}\mathrm{d}K(x-\xi)^{n}(\xi\,\theta\,y)^{n}[{}^{m}A_{\cdot}(\xi),\,{}^{n}F(y)] \quad \int {}^{m}\mathrm{d}K(x-\xi)^{m}(\xi\,|\,\theta\,|\,y)^{n}[{}^{m}A_{-}(\xi),\,{}^{n}F(y)]$$

and a similar equation involving B with identical content; the $(\pm)$ ordering is immaterial for the transform of the retarded commutator. Identities (26) may then by rewritten as

$$(30i)\qquad (-1)^{m}[{}^{m}\underset{i}{A}(x),\,{}^{n}F(y)] = \int {}^{m}\mathrm{d}K(x-\xi)^{m}(\xi\,|\,\theta\,|\,y)^{n}[{}^{m}A(\xi),\,{}^{n}F(y)]\,,$$

$$(30o)\qquad [{}^{m}F(x),\,{}^{n}\underset{o}{B}(y)] = \int {}^{n}\mathrm{d}K(y-\eta)^{m}(x\,|\,\theta\,|\,\eta)^{n}[{}^{m}F(x),\,{}^{n}B(\eta)]\,,$$

where we omitted the otiose subscripts of ${}^{m}A(\xi)$ and ${}^{n}B(\eta)$. This accords with the fact that $\underset{i}{A}$ and $\underset{o}{B}$ are unordered. Substituting ${}^{m}F(x) \to {}^{m}A_{-}(x)$, ${}^{n}F(y) \to {}^{n}B_{\pm}(y)$ into (30) and subjecting both sides to the K transform whose domains is y in $(30i)$ and x in $(30o)$, we obtain, with $(16b)$,

$$(31i)\qquad (-1)^{m}[{}^{m}\underset{i}{A}(x),\,{}^{n}\mathscr{B}_{\pm}(y)] = \int {}^{n}\mathrm{d}K(x-\xi)^{n}\,\mathrm{d}K(y-\eta)^{m}(\xi\,|\,\theta\,|\,\eta)^{n}\,[{}^{m}A(\xi),\,{}^{n}B)\eta)]\,,$$

$$(31o)\qquad [{}^{m}\mathscr{A}_{\pm}(x),\,{}^{n}\underset{o}{B}(y)] = \int {}^{m}\mathrm{d}K(x-\xi)^{n}\,\mathrm{d}K(y-\eta)^{m}(\xi\,|\,\theta\,|\,\eta)^{n}[{}^{m}A(\xi),\,{}^{n}B(\eta)]\,,$$

where the $(\pm)$ in the retarded commutator has been dropped by invoking (29).

The right-hand member of (31-i), identical with that of (31-o), and independent of the internal ordering of the A's and B's, plays a significant role in the next section. Of direct physical interest will be

$$(32)\qquad {}^{m}(x\,|\,\tau\,|\,y)^{n} = \int {}^{m}\mathrm{d}K^{-}(x-\xi)^{n}\,\mathrm{d}K^{+}(y-\eta)^{n}(\xi\,|\,\theta\,|\,\eta)^{n}\,\Omega_{-}[{}^{m}A(\xi),\,{}^{n}B(\eta)]\,\Omega_{-}\,.$$

This integral transform (with K^{-} on the initial, K^{+} on the final state) of the retarded commutator in the vacuum state is referred to as its projection unto the mass shall. Here we eliminate it between the (i) and (o) to obtain

two sets of reciprocity relations. The first,

$$\begin{cases} [{}^m\underset{i}{A}(x), {}^n\mathscr{B}_+(y)] = [{}^m\underset{i}{A}(x), {}^n\mathscr{B}_-(y)] , \\ [{}^m\mathscr{A}_+(x), {}^n\underset{o}{B}(y)] = [{}^m\mathscr{A}_-(x), {}^n\underset{o}{B}(y)] , \end{cases} \tag{33}$$

states that the unordered character of $\underset{i}{A}$ and $\underset{o}{B}$ renders the ordering of the radiation operators with which they are commuted immaterial. A single denotation in the commutator then does for both orderings. The second,

$$[{}^m\mathscr{A}(x), {}^n\underset{o}{B}(y)] = (-1)^m[{}^m\underset{i}{A}(x), {}^n\mathscr{B}(y)] , \tag{34}$$

relates the commutator of a radiation operator with an outgoing field to that of an incoming.

7. – Reduction of S.

With the material of the preceding three sections this is done quite simply We put

$$ {}^m(x\,|\,S\,|\,y)^n = \Omega_-\,{}^m\underset{i}{A}{}^-(x)\,{}^n\underset{o}{B}{}^-(y)\,\Omega_+ = \Omega_-[{}^m\underset{i}{A}{}^-(x), {}^n\underset{o}{B}{}^+(y)]\,\Omega_+ + \delta^{m0}\,\delta^{n0} , \tag{35}$$

where we added a Kronecker delta term to make the equality sign hold for all non-negative integers m, n. Substituting $(D_1 a)$ for $\underset{o}{B}$, we have

$$\begin{aligned} {}^m(x\,|\,S\,|\,y)^n = \frac{1}{n!}\sum_{\varrho}\binom{n}{\varrho}\sum_{q}\Omega_-[{}^m\underset{i}{A}(x), {}^{\varrho q}\underset{i}{B}{}^+(y)]\,{}^{\varrho q}\mathscr{B}'_+{}^+(y)\Omega_+ + \\ + \frac{1}{n!}\sum_{\varrho}\binom{n}{\varrho}\sum_{q}\Omega_-\,{}^{\varrho q}\underset{i}{B}{}^+(y)[{}^m\underset{i}{A}{}^-(x), {}^{\varrho q}\mathscr{B}'^+(y)]\,\Omega_+ + \delta^{m0}\,\delta^{n0} , \end{aligned} \tag{36}$$

where (33) was used to drop the $+$ ordering subscript from the $\mathscr{B}$ in the commutator of the second term of the right-hand member. We now identify $\underset{i}{A}, \underset{i}{B}$ with the free fields of E_- and use this relation to substitute for the commutator in the first term, with the simple result

$$ {}^n(nx\,|\,S\,|\,y)^n = \frac{1}{m!}\frac{1}{n!}\sum_{\sigma\varrho}\binom{m}{\sigma}\binom{n}{\varrho}\sum_{pq}{}^{\sigma p}(x\,|\,s\,|\,y)^{\varrho q}\Omega_-\,{}^{\sigma p}\underset{i}{A}{}'^-(x)\,{}^{\varrho q}\mathscr{B}'_+{}^+(y)\Omega_+ , \tag{37}$$

because we have

$$\begin{aligned} \frac{1}{n!}\sum_{\varrho}\binom{n}{\varrho}\sum_{q}\Omega_-\,{}^{\varrho q}\underset{i}{B}{}^+(y)[{}^m\underset{i}{A}{}^-(x), {}^{\varrho q}\mathscr{B}^+(y)]\,\Omega_+ + \\ + \delta^{m0}\,\delta^{n0} - \frac{1}{n!}\sum_{\varrho}\binom{n}{\varrho}\sum_{q}\delta^{\varrho 0}\Omega_-\,{}^m\underset{i}{A}(x)^{\varrho' q}\mathscr{B}'_+{}^+(y)\Omega_+ = 0 . \end{aligned} \tag{38}$$

The last term of (38) comes from the «boundary term» from E_-. To verify (38), observe that the only contribution to the ϱ sum of the first and last term is from $\varrho = 0$; the middle combines with the last to give a commutator that cancels the first. The A in (37) is now commuted with the $\mathscr{B}$ (with the addition of another Kronecker delta term) and this commutator $[A_i^-(x), \mathscr{B}^+(y)]$ is just the τ defined by (32). The expansion

$$(39a) \quad {}^m(x|S|s)^n = {}^m(x|s|y)^n + \frac{1}{m!\,n!}\sum_{\sigma\varrho}\sum_{p,q}(-1)^\sigma \binom{m}{\sigma}\binom{n}{\varrho}^{\sigma p}(x|\tau|y)^{q\varrho\sigma p'}(x|s|y)'^{q\varrho}$$

is obtained after some trivial changes in the summation variables. An alternative procedure is to first express the incoming field by means of $(D_2 b)$ in terms of the outgoing and then have recourse to E_+. In this scheme one has, in the final stage, vacuum expectation values of commutators of the type $[\mathscr{A}^-(x), B_\varrho^+(y)]$. Reciprocity (34) then indicates that the result is the same by virtue of an operator identity.

Without redundant sums and symmetrizations, (39a) states

$$(39b) \qquad {}^m(x|S|y)^n = {}^m(x|s|y)^n + (-1)^m \sum_{\sigma=0}^{\mu(m,n)} (-1)^\sigma \sum_{p|\sigma}\sum_{q|\sigma}{}^{\sigma p}(x|s|y)^{q\sigma\ \sigma p\ '}(x|\tau|y)'^{q\sigma},$$

where we make use of the orthogonality of initial and final states for the free s matrix. The μ associated with the upper limit of the summation means «the lesser of». Such explicit indications of limits are not needed in the representations with binomial coefficients. It is this accident that recommends them for certain tasks. Expression (39b) is, however, more convenient for generating particular expansion of S in terms of s. We see from (32) that we have ${}^0\tau^n = {}^m\tau^0 = 0$ for all m and n. As a test of the algebraic consistency of the formula, we also note that

$$(40o) \qquad {}^0(x|S|y)^n = \delta^{0n} \quad \text{and} \quad {}^m(x|S|y)^0 = \delta^{m0}$$

follow from this property of τ. For the simplest processes — decays — the expansions are

$$(40_1) \qquad \begin{cases} (x|S|y) = (x|s|y) - (s|\tau|y)\,, \\ (x|s|y_1 y_2 \ldots y_n) = -(x|\tau|y_1 y_2 \ldots y_n) \qquad \text{for } n > 1\,. \end{cases}$$

With two fields in the initial state the various possibilities in the final are

indicated in

$$(40_2)\quad \begin{cases} (x_1x_2|S|y) = (x_1x_2|\tau|y)\,, \\ (x_1x_2|S|y_1y_2) = (x_1x_2|s|y_1y_2) + (x_1x_2|\tau|y_1y_2) - (x_1|s|y_1)(x_2|\tau|y_2) - \cdots, \\ (x_1x_2|S|y_1y_2\ldots y_n) = (x_1x_2|\tau|y_1y_2\ldots y_n) - (x_1|s|y_1)(x_2|\tau|y_2\ldots y_n) - \cdots \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } n > 2\,. \end{cases}$$

The presence of one-to-one particle-transition terms should be noted. They do not appear in a properly renormalized theory that deals only with stable particles.

The combinational sense of the expansion is evident. A game which m particles enter and n leave is played according to all possible arrangements of spectators and participants. Throughout each engagement (each term of the expansion) the spectators maintain their number and identity; players may be lost in or new ones emerge from the scuffle. Since preponderantly spectator sports do not seem to be popular at high energies, the expansion does not signify.

We now discuss the relation of the reduction (39) to other reductions of the S matrix. The idea is an old one; its nearest classical correlative is, perhaps, found in the work of Dirac. The energy (power) radiated by a charge distribution may be calculated from the time integral (average for discrete lines) of the surface integral of the Poynting vector at a surface remote from the source. By defining a radiation field inside the charge distribution this author was able to transform the asymptotic representation into a space-time integral over the charge. This operation corresponds to $\mathrm{d}H \to \mathrm{d}K$ integration. Most of the perturbation treatments of scattering that favor the Heisenberg picture follow this impulse of Dirac rather closely. They are equivalent to the use of (D_2a) to obtain

$$(41i)\qquad {}^{m}(x|S|y)^{n} = \frac{1}{n!}\sum_{\varrho}\binom{n}{\varrho}\sum_{q}\Omega_{i}^{-\,m}(x)\,{}^{\varrho q}\mathscr{B}_{-}^{+}(y)\,{}^{\varrho q}\Omega_{i}^{\prime\,+}(y)\,,$$

where the expectation of the $(-)$ ordered radiation fields (earlier times to the right) is taken in states constructed of incoming fields. The observation that the other choice would be just as good is equivalent to the use of (D_1b), which gives

$$(41o)\qquad {}^{m}(x|S|y)^{n} = \frac{(-1)^m}{m!}\sum_{\sigma}\binom{m}{\sigma}(-1)^{\sigma}\sum_{p}\Omega_{o}^{p\sigma}(x)\,{}^{\sigma' p}\mathscr{A}_{-}^{\prime\,-}(x)^{n}\Omega_{o}^{\prime\,+}(y)\,.$$

The joint use of (D_2a) and (D_1b) produces a complete reversal of fields:

$$(41r)\qquad {}^{m}(x|S|y)^{n} = \frac{(-1)^m}{m!\,n!}\sum_{\sigma\varrho}(-1)^{\sigma}\binom{m}{\sigma}\binom{n}{\varrho}\sum_{pq}\Omega_{o}^{p\sigma}(x)\,{}^{q' p}\mathscr{A}_{-}^{\prime\,-}(x)\,{}^{\varrho q}\mathscr{B}_{-}^{+}(y)\,{}^{\varrho' q}\Omega_{i}^{\prime\,+}(y)\,.$$

We are not aware of any past use of this representation. It should be noted that the $(\pm)$ order is pertinent to (41). Somewhat outside all this is the Green's-function formalism of Schwinger. Its reflection in the S-matrix context amounts to an indiscriminate single ordering of all fields, initial and final. What distinguishes the reduction used in dispersion work from the others is the reciprocity (34) that enables us to omit the $(\pm)$ subscript; it is, on the other hand, decidedly emphatic in its discrimination between initial and final states.

The development of this article followed, too, the path of Dirac, but had a particular orientation: it sought to express S in terms of s. Prompted by the simple observation that a matrix element of s — a c-number — could be identified with a commutator — a function on q numbers — of multiple fields, it proceeded to generate the former by means of the latter (E). To contrive a mutual confrontation in S of the two factors of this commutator it augmented the number of Dirac identities ($D_1 b$, $D_2 a$) that may be used in perturbation treatments with two more ($D_1 a$, $D_2 b$). The elements of τ were generated in the course of this confrontation.

8. – Microcausality.

We assume the existence of a complete,

$$\sum_{\alpha}\int \mathrm{d}\varkappa^2 \varrho(\varkappa, \alpha)\int \frac{\mathrm{d}^4 k}{(2\pi)^4}\, \Delta^+(k, \varkappa)\, \omega_+(k, \alpha)\, \omega_-(k, \alpha) = 1\,, \tag{42}$$

orthonormal,

$$\varrho(\varkappa, \alpha)\, \Delta^+(k, \varkappa)\, \omega_-(k, \alpha)\, \omega_+(k', \alpha')\, \Delta^+(k', \varkappa')\, \varrho(\varkappa', \alpha') = \tag{43}$$

$$= (2\pi)^4 \varrho(\varkappa, \alpha)\, \Delta^+(k, \varkappa)\, \delta(k - k')\, \delta(\varkappa^2 - \varkappa'^2)\, \delta_{\alpha\alpha'}\,,$$

set of eigenstates $\omega(k, \alpha)$ of the Hilbert-space representation U^D of the displacement group D,

$$U_a{}^D\, \omega_+(k, \alpha) = \exp\,[-\,ika]\, \omega_+(k, \alpha) \tag{44}$$

undistorted by mappings of the Lorentz group

$$\omega(k, \alpha) \underset{A(t)}{\to} \omega^t(k, \alpha) \equiv U_t{}^A \omega(k_t, \alpha) = \omega(k, \alpha)\,. \tag{45}$$

Completeness (42) and orthonormality (43) have been stated in terms of two distributions, rather than proper functions, in order to evidence their cova-

riant character. These are $\varrho(\varkappa, \alpha)$, (the spectral density of the masses) and

$$\Delta^+(k, \varkappa) = 2\pi\theta(k)\,\delta(k^2 - \varkappa^2)\,,$$

the Fourier transform of the familiar $\Delta^+(x)$. All other quantum numbers needed for the definition of a state have been designated by α. Orthogonality in these is indicated by a dimensionless Kronecker delta to accord with the fact that in (42) ω must have the dimension of length. Assumptions (42) and (43) imply the possibility of expanding an arbitrary element $\Omega_\pm$ of the Hilbert space

$$\Omega = \sum_\alpha \int \mathrm{d}\varkappa^2 \int \frac{\mathrm{d}^4k}{(2\pi)^4}\, \omega(k, \alpha)\ \Delta^+(k, \varkappa)\ \varrho(\varkappa, \alpha)\ \ \psi(k, \alpha)\,, \tag{46a}$$

with

$$\varrho(\varkappa, \alpha)\,\Delta^+(k, \varkappa)[\psi(k, \alpha) - \omega_-(k, \alpha)\,\Omega_+] = 0\,, \tag{46b$_+$}$$

$$\varrho(\varkappa, \alpha)\,\Delta^+(k, \varkappa)[\psi(k, \alpha) - \Omega_- - \omega_+(k, \alpha)] = 0 \tag{46b$_-$}$$

for contravariant and covariant Hilbert-space vectors, respectively.

Expanding ${}^m\!A_\pm(x)\omega_+(k, \alpha)$ and $\omega_-(k, \alpha){}^m\!A_\pm(x)$, we denote the vacuum components of these vectors, the Bethe-Salpeter amplitudes [4], by

$${}^m\psi^+_\pm(x;\, k, \alpha) = \omega_-(k, \alpha)\ {}^m\!A_\pm(x)\,\omega_+\,, \tag{47$_+$}$$

$${}^m\psi^-_\pm(x;\, k, \alpha) = \omega_-\ {}^m\!A_\pm(x)\,\omega_+(k, \alpha)\,. \tag{47$_-$}$$

Under a Lorentz mapping we have

$${}^m\psi^-_+(x\ k, a) \underset{A(t)}{\rightarrow} {}^m\psi_+{}^{t-}(x, k, \alpha) \equiv {}^m\psi^-_+(x_{-t}, k_t, \alpha)\,,$$

and with (47$_-$) and (45) obtain

$${}^m\psi^{t-}{}_+(x, k, \alpha) = \omega_- U_t{}^{-1m} A(x_{-t})\ U_{-t}{}^A\omega_+(k, \alpha)\,. \tag{48}$$

It is evident from (48) that ψ's admit the Lorentz group trivially, for $m=0$; by virtue of the defining equation (6) of the U^{-1} representation, for $m = 1$ but, in general, not for higher m, we can write

$${}^m\psi^t_\pm = {}^m\psi_\pm \qquad \text{for } m = 0,\ 1\ (A). \tag{49}$$

[4] K. Nishijima: *Progr. Theor. Phys.* (*Kyoto*), **12**, 279 (1954).

For the displacement group, because the fields are ordered relative to each other, we have, for all m,

$$U_a^D \, {}^m A_\pm(x-a) \, U_{-a}{}^D = {}^m A_\pm(x)$$

and therefore

$$ {}^m\psi_\pm^a = {}^m\psi_\pm \qquad \text{for all } m \quad (D) \tag{50}$$

To obtain a differential version of (50) we define

$${}^m\Box_\mu = \frac{\hat{c}}{\hat{c}x_1^\mu} + \frac{\hat{c}}{\hat{c}x_2^\mu} + \cdots + \frac{\hat{c}}{\hat{c}x_m^\mu} ,$$

and deduce from (47), (48), and the well-known exponential representation of U^D

$$i \, {}^m\Box_\mu \varrho(\varkappa\alpha) \, \Delta^+(k\varkappa) {}^m\psi^\pm(x, k\alpha) = \pm \, k_\mu \varrho(\varkappa\alpha) \, \Delta^+(k, \varkappa) \, {}^m\psi^\pm(x, k\alpha) , \tag{51}$$

which indicates that $\varrho \, \Delta^+ \psi^+$ and $\varrho \, \Delta^+ \psi^-$ are the positive and negative frequency functions satisfying the generalized Gordon-Klein equation

$$({}^m\Box^2 + \varkappa^2) \, \varrho(\varkappa, \alpha) \, \Delta^+(k, \varkappa) \, {}^m\psi_\pm(x, k, \alpha) = 0 . \tag{52G_0}$$

The definition of τ may now be given in terms of the B^-S amplitudes:

$${}^m(x \,|\, \tau \,|\, y)^n = \int {}^m\mathrm{d}K^-(x-\xi)^n \, \mathrm{d}K^+(y-\eta)^m \, (\xi \,|\, \theta \,|\, \eta)^n \, {}^m(\xi \,|\, D \,|\, \eta)^n , \tag{53}$$

$${}^m(x \,|\, D \,|\, y)^n = \sum_\alpha \int \mathrm{d}\varkappa^2 \, \varrho(\varkappa, \alpha) \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \, \Delta^+(k, \varkappa) \cdot \tag{54}$$

$$\cdot \left[{}^m\psi^-(x, k, \alpha)^n \varphi^+(y, k, \alpha) - {}^m\psi^+(x, k, \alpha) \, {}^n\varphi^-(y, k, \alpha) \right] ,$$

where ψ pertains to the initial, φ to the final state. The $(\pm)$ ordering subscripts of ψ and φ have been omitted in accordance with (32). For the special case $m = n = 1$, this reduces to the well-known bilinear representation of the radiation kernel, vanishing for $(x-y)^2 < 0$ and admitting the Lorentz group. The latter property is an immediate consequence of (49), and because of the restricted validity of this equation, need not hold for other m and n. The implications of the requirement that S admit the Lorentz group are now plain. Since s admits the group, this burden is shifted to τ. An inspection of (53) indicates that the radiation kernel D with an arbitrary number of initial and final particles must conform in its basic properties to that of a

single particle:

(*a*) ${}^{m}(x|D|y)^{n}$ must admit the Lorentz group for all m, n;

(*b*) ${}^{m}(x|D|y)^{n} = 0$ if $(x_i - y_j)^2$ is less than 0 for any $i = 1 \ldots m, \quad j = 1 \ldots n$.

That the physically significant quantity, τ, evinces no interest in our nice distinctions between the + and — order is suggestive. Let us imagine that we expanded the single equation $(52G_0)$ into an set $(G_0, G_1, \ldots G_s \ldots a_{\pm})$, whose solutions are ψ's equivalent to the definition (47) of ψ in terms of expectation values of ordered operators. The G's of this set are differential equations coupling the various amplitudes, and the $a_{\pm}$ denotes statements of asymptotic conditions, which descriminate between the orderings. Since they are irrelevant to τ, one is tempted to replace them by a set (b) of boundary conditions; that is, statements of the asymptotic behavior of the solutions ${}^{m}\psi(x)$ of the system $(G_0', \ldots G_s' \ldots b)$ for large specelike separations of the arguments. The primes of the G indicate that these boundary conditions might be more simply expressed for certain linear combinations of ψ (the Tamm-Dancoff wave functions, say, rather than the Bethe-Salpeter amplitudes) [4]. It is not unreasonable to expect that these boundary conditions could be satisfied only for certain values of $\varkappa$ and thus determine $\varrho(\varkappa)$ in terms of a few elementary masses. The wave functions obtained from solving this set could also be used to construct the bilinear form (54).

This suggests the view that it is the wave mechanics $(G'b)$ that is the substantial part of quantum theory of fields; manipulations with operators are merely an umbral calculus to generate the mechanics. This descriptive term, borrowed from combinational analysis, implies there that the analytic behavior of the generating functions is irrelevant to the final result, the correct enumeration of things. One might perhaps adopt a similar view that the precise conditions imposed on field operators are not particularly germane to the substantial part of field theory. This mechanics finds its inchoate expression, at the moment, in dispersion theory. In it, an attempt is made to exploit one of the facets of the mechanics, the vanishing of a bilinear form on the wave functions in certain regions and the existence of a mass spectrum with positive frequencies.

9. — Summary.

An arbitrary element of S may be expanded in terms of elements of s with elements of τ as coefficients.

The elements of τ are mass-shell projections of a retarded generalized Green's function which is a product of a retardation factor ordering the final relative to the initial state, and a generalized radiation kernel D.

The kernel D is a bilinear form in the BS amplitudes, appropriate to the initial and final states.

The internal ordering of the fields of the initial and final states entering into the definition of the BS amplitudes in terms of matrix elements of products of operators is immaterial for D.

The requirement that S follow the example set by s in admitting the Lorentz group (causality) leads to the condition that ${}^{m}D^{n}$ conform to the pattern established by the single-field kernel ${}^{1}D^{1}$ in admitting the group and in vanishing for spacelike separations of the initial and final states (microcausality).

* * *

The author enjoyed a conversation with Dr. Geoffrey Chew regarding matters treated in this article.

RIASSUNTO (*)

Si introduce una definizione della causalità, differente da quella adoperata usualmente nella teoria dei campi. Si pone in chiaro la sua relazione con il gruppo di Lorentz e con la teoria della relatività. Essa porta ad una riduzione della matrice S, adatta alla teoria della dispersione, che elimina alcune delle limitazioni incontrate sinora. Si discutono le formule matematiche che compaiono nel corso di questa riduzione.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XIX, N. 3 1° Febbraio 1961

$\pi^- + p$ Elastic Scattering at 1200 MeV.

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(ricevuto il 23 Luglio 1960)

Summary. — A bubble chamber investigation of $\pi^- + p$ elastic scattering at 1200 MeV (K.E.) is reported. The total and differential cross-sections are determined. By extrapolation of the angular distribution, the 0° cross-section is derived and compared with the results obtained with the help of the dispersion relations and the optical theorem. The forward peak is investigated in terms of diffraction scattering and a value for the optical radius is derived.

1. – Introduction.

1·1. – In this paper we report some experimental results concerning an investigation of $\pi^- - p$ elastic scattering in a propane bubble chamber (~ 30 cm in diameter and ~ 21 cm in depth, provided with a 13.4 kG magnetic field) exposed to a 1200 MeV (K.E.) π beam at the Brookhaven cosmotron (**).

In Section 2 the method used in the analysis is presented and the problem of carbon contamination is studied. In Section 3 the angular distribution and the total cross-section are discussed.

(*) On leave of absence from the Poftchestroom University, South Africa.

(**) For details on the arrangement of beam and exposure see F. Eisler *et al.* (1).

(1) F. Eisler, R. Plano, A. Prodell, N. Samios, M. Schwartz, P. Bassi, V. Borelli, G. Puppi, H. Tanaka, P. Waloschek, V. Zoboli, M. Conversi, P. Franzini, I. Mannelli, R. Santangelo and G. V. Silvestrini: *Nuovo Cimento*, **10**, 468 (1958).

2. – Method of analysis.

2·1. *Beam energy.* – π^--p elastic scatterings appear as two-prong stars produced by the incident mesons, and must be separated from the other events found in the chamber. For this purpose it is useful to have a careful determination of the pion beam energy. This was determined from a study of 35 events of charged strange particles (Σ^-, K^-) production, which were obtained in the same exposure. From the kinematical data on these events, the mean energy of the incident pions was found to be (1204 ± 8) MeV. The energy spread was limited by the collimation system to $\lesssim 1\,\%$. These figures are confirmed by the present analysis of the elastic scattering events.

2·2. *Fiducial region.* – Since the detection efficiency of the events depends on the length of the tracks and therefore on the position in the chamber, we have chosen in it a fiducial region and have based all results only on the events whose vertices fall within the acceptance volume. The minimal potential path for a track was required to be 8 cm (in the 1.857 magnified reprojection).

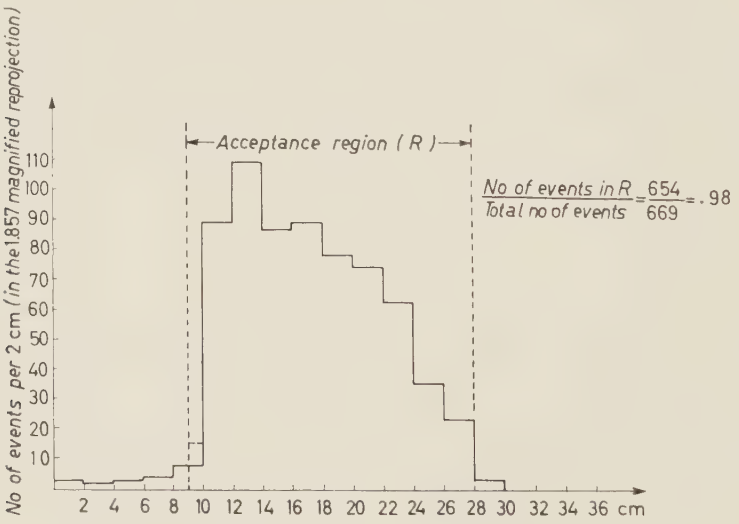

Fig. 1*a*. – Distribution in depth of 669 elastic events.

In Fig. 1*a* and 1*b* the distributions in depth of 669 elastic events and of 552 stars are reported.

The detection efficiency seems uniform in the volume of the fiducial region.

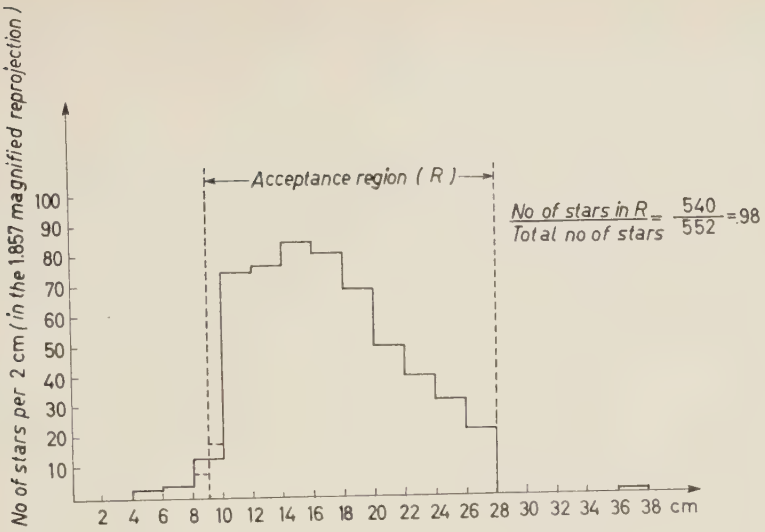

Fig. 1*b*. – Distribution in depth of 552 stars.

2·3. *Selection criteria.* – A star with two prongs of opposite charges is accepted as an elastic scattering by a free proton when the following conditions are satisfied within the measurement errors (which depend on the length and orientation of the tracks, on the plane of the event, ecc.).

1) It must be coplanar.

2) *a*) the angles θ_π and θ_p which the directions of the scattered pion and the recoil proton form with the direction of the incident meson;

b) the momenta of the emerging particles calculated by magnetic curvature measurements;

c) the ranges of the protons which stop inside the chamber (*), must be consistent with the kinematics of the process.

3) Bubble density and multiple scattering must be qualitatively consistent with the expected ones.

2·4. *Contamination.* – The events which we have thus selected include also a contamination of spurious events—such as carbon events or hydrogen inelastic events—which simulate the elastic hydrogen scattering. The observed distributions of the relevant dynamical parameters referred both to the « elastic » as well as to the « rejected » events gave us a method with which to calculate this contamination.

(*) At this energy elastically scattered pions do not stop inside the chamber.

In order to do that we have selected at random a sample of 532 two-branch stars, with the requirement that the two secondary particles had opposite charges.

For each event we evaluated:

a) the « non-coplanarity », defined as the angle η between the direction of the incident particle and the plane of the emerging ones; *b*) the minimal distance d in degrees between the point representing each event in the plane θ_π, θ_p and the curve, which correlates these angles for the π^--p elastic scattering at the given energy (Fig. 2). In Fig. 3 and 4 the distributions *vs.* η

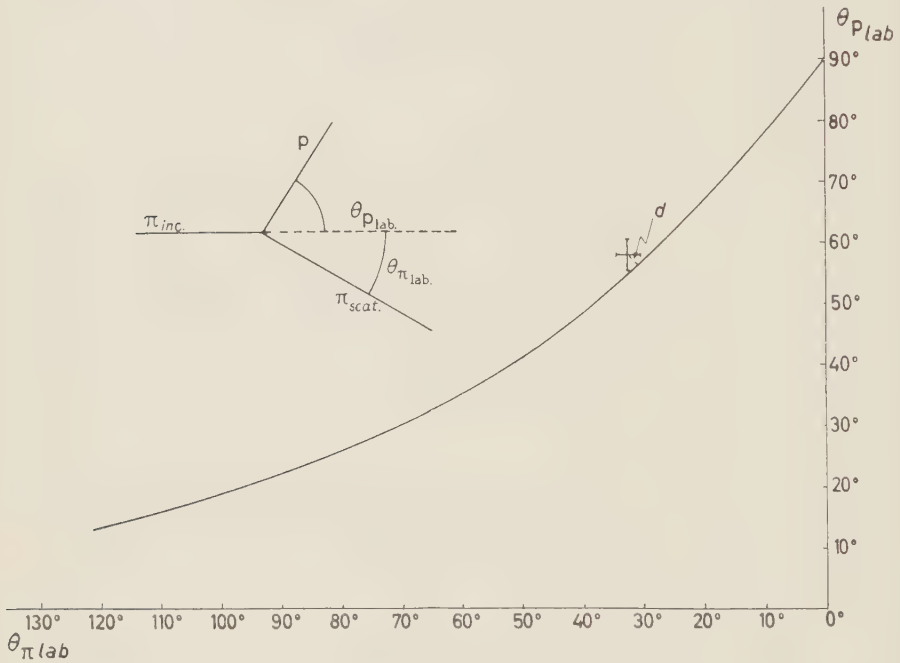

Fig. 2. – A portion of the kinematic curve giving the recoil proton angle as a function of the pion scattering angle for incoming pions of 1200 MeV (K.E.).

and d of the elastic events are given. It is seen that 156 and 165 of the 174 elastic events occur respectively in the intervals $0° \leqslant \eta \leqslant 2°$ and $0° \leqslant d \leqslant 2°$ and that 166 and 173 occur in the intervals $0° \leqslant \eta \leqslant 3°$ and $0° \leqslant d \leqslant 3°$. The corresponding distributions for the rejected events are given in Fig. 5 and 6.

In the same intervals, *e.g.*

(α) $\qquad 0° \leqslant d\,, \qquad \eta \leqslant 2°$

or

(β) $\qquad 0° \leqslant d\,, \qquad \eta \leqslant 3°$

also some of the rejected events fall. Let d_α be the number of rejected events satisfying condition (α) for the d parameter; and η_α the analogous number

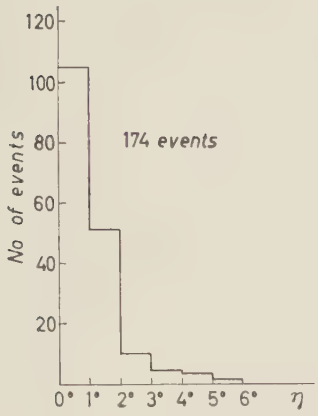

Fig. 3. – Distribution *vs.* « non-coplanarity » η of the elastic events.

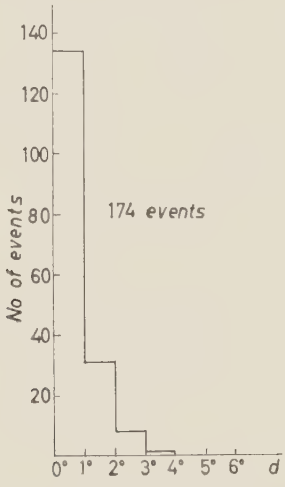

Fig. 4. – Distribution *vs.* d of the elastic events.

for η. If R is the total number of rejected events, then

$$P_\alpha = \frac{\eta_\alpha}{R} \cdot \frac{d_\alpha}{R}$$

Fig. 5. – Distribution *vs.* « non-coplanarity » η of 358 rejected events.

can be taken as the probability that a non-elastic event satisfies condition (α). Similarly one can reason for condition (β).

Introducing the numbers thus found (see Fig. 5 and 6) we obtained

$$(\alpha') \qquad P_\alpha = \frac{72}{358} \cdot \frac{40}{358} ,$$

in case (α), and

$$(\beta') \qquad P_\beta = \frac{93}{358} \cdot \frac{60}{358}$$

in case (β) (*).

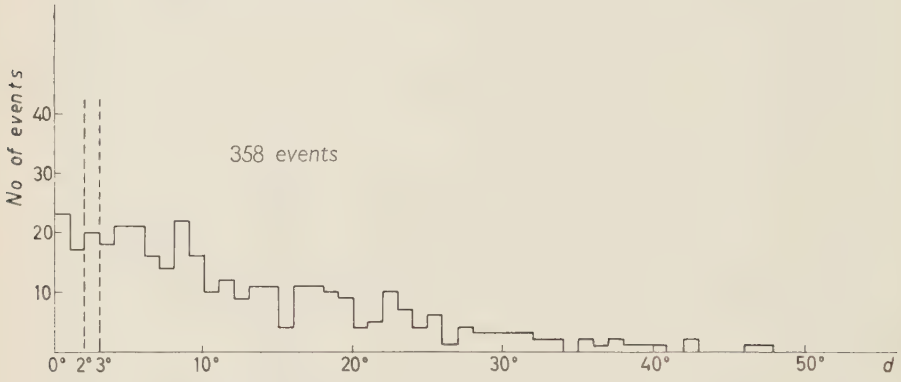

Fig. 6. – Distribution *vs.* d of 358 rejected events.

From (α') and (β') we obtain a contamination of 8 and 15.6 events respectively. By subtraction of the number of events satisfying criteria 1) and 2*a*) but rejected by means of the other criteria, we obtain a net number of 2 events in case (α) and 2.6 in case (β). These figures correspond to contaminations of $(1.2 \pm 1.7)\%$ in case (α) and $(1.5 \pm 2.6)\%$ in case (β), which coincide within the errors. The last result will be taken in the following as a measure of contamination.

2·5. *Scanning bias.* – The detection efficiency is uniform in the acceptance volume, but depends on the features of the event, *e.g.* events with a short recoil are hardly recognized. For this reason we have rejected all the events with a recoil corresponding to a C.M. scattering angle smaller than arcos .965.

(*) Actually d_α/R and η_α/R are not necessarily independent and the results are not entirely justified. Anyway, for the evaluation of contamination some assumption or extrapolation must be made, and because of the low value of the contamination a more rigorous calculation is not required.

The detection efficiency depends also on the orientation of the plane of the event, *i.e.* on the angle φ between this plane and a plane parallel to the glass windows. In Fig. 7 the distributions of the elastic events *vs.* φ for the

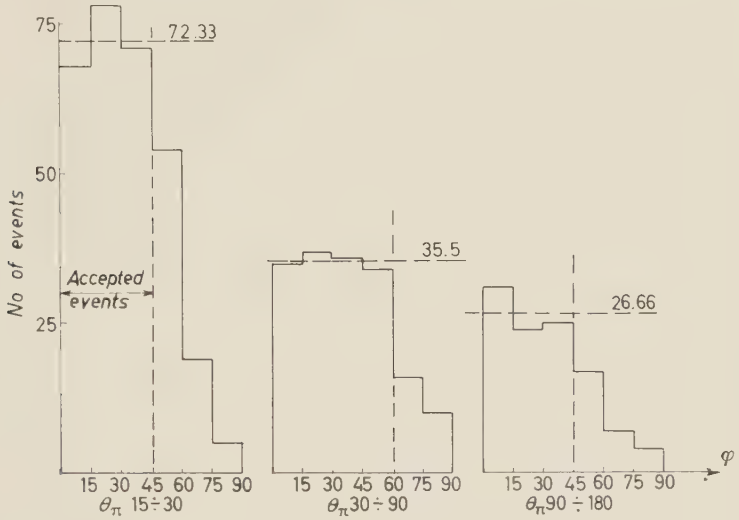

Fig. 7. – Distribution of the elastic events, *vs.* φ (azimuthal angle of the plane of the events around the line of flight of the incident track), for three C.M. scattering angle intervals.

C.M. scattering angle intervals are reported. Only the ranges of φ in which they are approximately uniform have been considered for the results. In this manner we disregard a number of events (which generally are also not well measurable), but we have more confidence in the results. Accordingly, a correction factor (2; $\frac{3}{2}$; and 2 in the three cases shown) will be included in the final results. Another slight correction is derived in the usual way by comparison between the number of events detected by different scanners.

2·6. *Total track length.* – To determinate the total track length, we measured the projected lengths of all the « beam » tracks inside the acceptance region in every 100-th picture. Only tracks, forming with the mean direction of the beam a projected angle α smaller than 1°.5 have been counted. The distribution *vs.* α of 193 accepted tracks is given in Fig. 8-*a*). The corresponding distribution for 491 tracks of particles suffering an elastic scattering is given in Fig. 8*b*). The dip angle of every beam track with α larger than 0°.5 has been measured and only the tracks, for which it was found to be 0° within the errors, have been counted. The total path length turns out to be

$(30.47 \pm .79) \cdot 10^5$ cm. From this number the muon and electron track lengths have to be subtracted. The muon contamination is estimated as $(6 \pm 4)\%$ [2] the electron contamination has been ignored [2]. The corrected track length is then determined as $L = (28.64 \pm 1.37) \cdot 10^5$ cm.

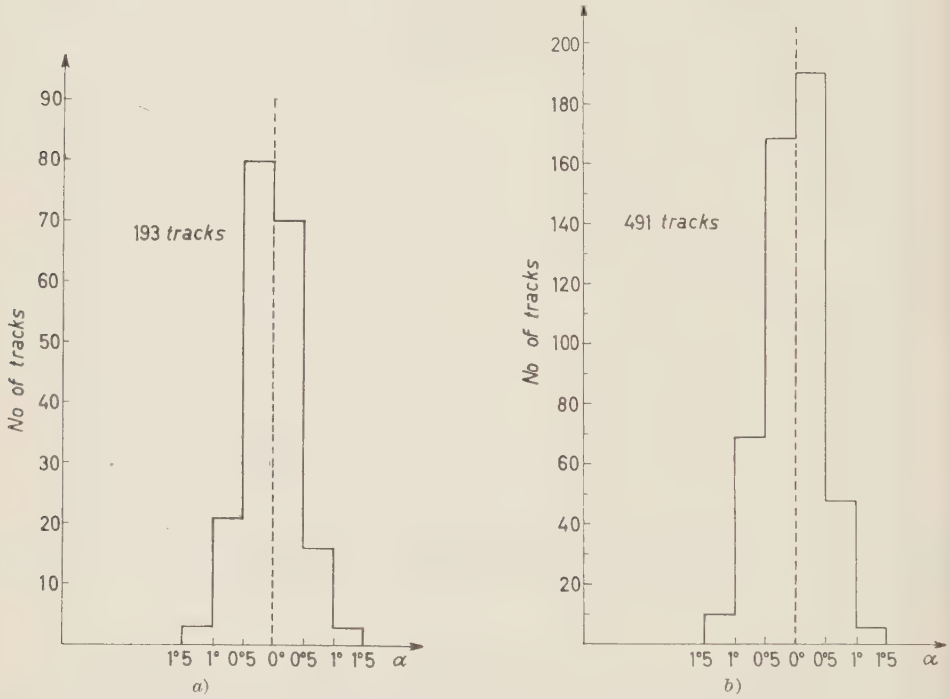

Fig. 8. – *a*) Distribution of 193 beam tracks (not suffering scattering) *vs.* α (projected angle of a track with the mean direction of the beam). *b*) Distribution *vs.* α of 491 tracks of particles suffering an elastic scattering.

3. – Results.

3·1. *Angular distribution.* – In Table I the numbers of observed events in 23 intervals of $\cos\theta_{\text{C.M.}}$ and the corresponding numbers corrected for the scanning bias are reported together with their statistical errors. Fig. 9 gives the corrected angular distribution, normalized to 12 mb (as it will be determined in Section 3·2).

It may be written as

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_{\text{C.M.}} = \sum_{i=0}^{n} a_i \cos^i \theta_{\text{C.M.}} ,$$

[2] R. Cool, O. Piccioni and D. Clark: *Phys. Rev.*, **103**, 1082 (1956).

TABLE I.

$\cos\theta_{C.M.}$	Intervals of $\cos\theta_{C.M.}$	N_1	Accepted intervals of φ	N_2	N	Corrected errors
.932 5	.965 ÷ .900	185	0° ÷ 45°	569.2	572.9	42
.875	.900 ÷ .850	95	»	380	382.5	39.1
.825	.850 ÷ .800	62	»	248	249.6	31.6
.775	.800 ÷ .750	43	»	172	173.1	26.3
.725	.750 ÷ .700	35	»	140	140.9	23.8
.675	.700 ÷ .650	35	»	140	140.9	23.8
.625	.650 ÷ .600	18	»	72	72.5	17.1
.550	.600 ÷ .560	6	»	30	30.2	7.1
	.600 ÷ .500					
	.560 ÷ .500	12	0° ÷ 60°			
.450	.500 ÷ .400	15	»	22.5	22.6	5.8
.350	.400 ÷ .300	5	»	7.5	7.5	3.4
.250	.300 ÷ .200	2	»	3	3	2.1
.150	.200 ÷ .100	9	»	13.5	13.6	4.5
.050	.100 ÷ .000	10	»	15	15.1	4.7
.050	.000 ÷ — .100	21	»	31.5	31.7	6.9
.150	— .100 ÷ — .200	18	»	27	27.2	6.3
.250	— .200 ÷ — .300	19	»	28 5	28.7	6.5
.350	— .300 ÷ — .400	13	»	19.5	19.6	5.4
.450	— .400 ÷ — .500	14	»	21	21.1	5.6
.550	— .500 ÷ — .600	14	»	21	21.1	5.6
.650	— .600 ÷ — .700	26	0° ÷ 45°	52	56.7	10.8
.750	— .700 ÷ — .800	20	»	40	43.6	9.3
.850	— .800 ÷ — .900	19	»	38	41.5	9.1
.950	— .900 ÷ — .1	21	»	42	45.8	9.6

N_1: real no. of events.
N_2: no. of events per $\Delta\cos\theta_{C.M.} = .1$ and $\Delta\varphi = 90°$.
N : no. of events per $\Delta\cos\theta_{C.M.} = .1$ and $\Delta\varphi = 90°$ corrected for detection efficiency.

where the coefficients a_i are determined by a least squares method and the polynomial degree n by a « goodness test » of the fits with various n. The

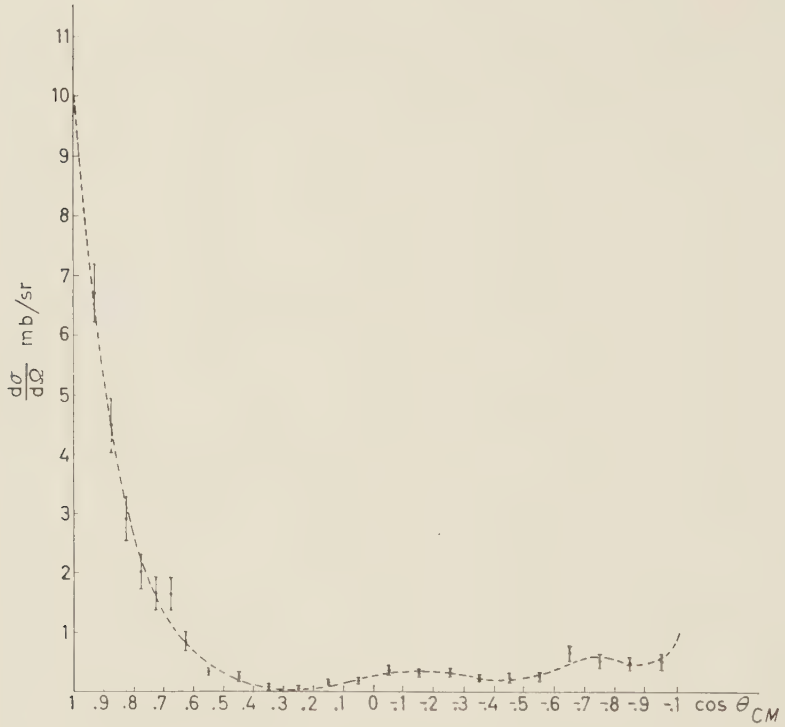

Fig. 9. – Corrected differential cross-section.

« goodness test » is made by plotting the quantity

$$(\gamma) \qquad M = \sqrt{\sum_{r=1}^{N} \frac{[Y(x_r) - Y_r]^2}{P(x_r)[N-(n+1)]}}$$

vs. the polynomial degree n, and looking at its minimal value. In formula (γ) $Y(x_r)$ are the values of the polynomial for $\cos\theta_{\text{C.M.}} = x_r$, Y_r the corresponding measured values, $P(x_r)$ the weight assigned to each experimental point (*), N the number of the intervals in which the range $(.965 \div 1)$ for $\cos\theta_{\text{C.M.}}$ has been divided. In Table II the coefficients a_i, their errors (**), and the values

(*) In each interval, the weights have been assumed proportional to the corresponding inverse squared errors.

(**) The statistical errors and the errors in the determination of the detection efficiency have been considered for this calculation.

of M (in arbitrary units) are given for various n. The minimum value of M is obtained for $n = 9$; the corresponding polynomial is reported in Fig. 9 together with the experimental results.

TABLE II.

n	a_i (mb/sr)	M	n	a_i (mb/sr)	M
6	$.23 \pm .04$	1.287	9	$.27 \pm .04$	1.024
	$.48 \pm .15$			$-1.01 \pm .25$	
	$-1.31 \pm .63$			$-2.34 \pm .90$	
	$.89 \pm .93$			9.65 ± 3.71	
	6.10 ± 2.23			13.26 ± 5.73	
	3.43 ± 1.12			-31.87 ± 16.93	
	$-.70 \pm 2.01$			-17.90 ± 12.24	
				47.99 ± 28.75	
7	$.23 \pm .04$	1.197		12.31 ± 8.09	
	$-.75 \pm .20$			-20.19 ± 16.10	
	$-.88 \pm .63$				
	4.16 ± 1.95		10	$.26 \pm .04$	1.052
	3.76 ± 2.42			$-1.00 \pm .25$	
	-6.10 ± 5.20			-1.73 ± 1.41	
	1.74 ± 2.29			9.08 ± 3.94	
	7.39 ± 3.95			6.44 ± 13.35	
				-28.03 ± 18.67	
8	$.27 \pm .04$	1.045		8.37 ± 47.87	
	$-.79 \pm .18$			40.05 ± 32.69	
	$-2.58 \pm .90$			-27.34 ± 70.21	
	5.56 ± 1.80			-15.21 ± 18.73	
	15.79 ± 5.48			20.21 ± 35.54	
	-11.57 ± 5.09				
	-24.61 ± 11.24				
	12.27 ± 4.01				
	17.22 ± 7.22				

In Table III some values of Y (for $n = 9$) and the corresponding errors are given. From these data we can calculate the laboratory cross-section at 0°

TABLE III

$x = \cos\theta_{\text{C.M.}}$	$Y(x)$	$x = \cos\theta_{\text{C.M.}}$	$Y(x)$
1	10.16 ± 1.66	0	$.03 \pm .04$
.950	$7.59 \pm .66$	$-.200$	$.33 \pm .04$
.850	$3.80 \pm .23$	$-.400$	$.20 \pm .04$
.700	$1.26 \pm .12$	$-.600$	$.38 \pm .06$
.500	$.32 \pm .05$	$-.800$	$.55 \pm .08$
.300	$.06 \pm .02$	-1	$1.04 \pm .45$

which results

$$\left(\frac{\mathrm{d}\sigma(0^\circ)}{\mathrm{d}\Omega}\right)_{\mathrm{lab}} = (39.4 \pm 6.4)\ \mathrm{mb/sr}\,.$$

The absolute error would be increased to $\pm$ 6.8 mb/sr by taking into account the errors in the total path length, density and contamination. In Fig. 10 the determined forward cross-section is compared with the results obtained

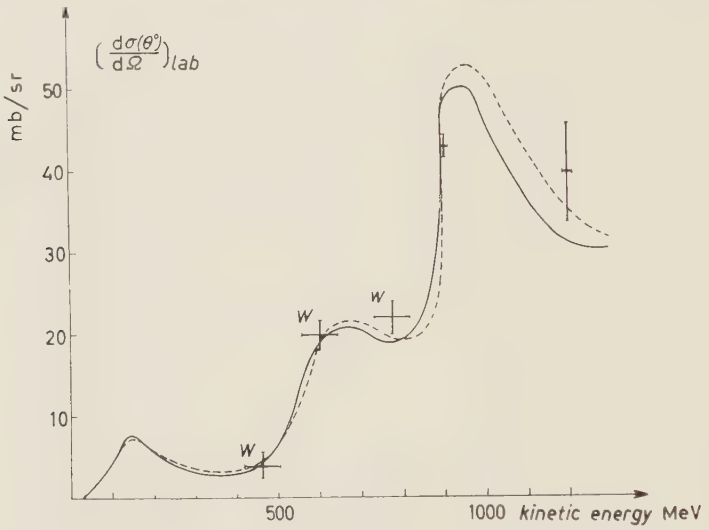

Fig. 10. – Lab. forward cross-section as a function of the incoming pion energy. The points « W » were obtained by S. Bergia *et al.* [3] extrapolating the data of R. R. Crittenden *et al.* [4].

by S. Bergia *et al*, [3] with the help of the dispersion relations assuming for the π-p interaction coupling constant the values $f^2 = .08$ and $f^2 = .1$.

If the elastic scattering could be interpreted as only shadow scattering, the angular distribution would be given by

$$(\delta) \qquad \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \propto \left| \frac{J_1(KR \sin\theta_{\mathrm{C.M.}})}{KR \sin\theta_{\mathrm{C.M.}}} \right|^2 ,$$

where K and R are respectively the pion wave number in the C.M. system and the optical radius.

[3] S. Bergia, L. Bertocchi, V. Borelli, G. Brautti, L. Chersovani, L. Lavatelli, A. Minguzzi-Ranzi, R. Tosi, P. Waloschek and V. Zoboli: *Nuovo Cimento*, **15**, 551 (1960).

[4] R. R. Crittenden, J. H. Scandrett, W. D. Shephard, W. D. Walker and J. Ballam: *Phys. Rev. Lett.*, **2**, 121 (1959).

We do not expect equation (δ) to fit the experimental data, precisely except perhaps at the small angles, where the contribution of the diffraction scattering is probably high. Fig. 11 gives curve (δ) for $R = 1.08 \cdot 10^{-13}$ cm (the same value as obtained by M. CHRÉTIEN *et al.* [5] at 1300 MeV).

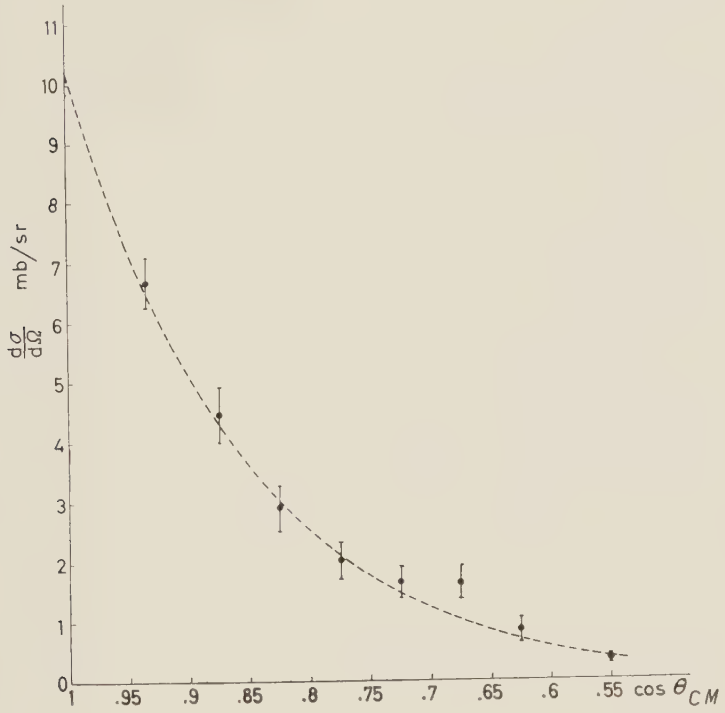

Fig. 11. – Differential cross-section at small angles. The dashed line is a black-sphere diffraction pattern for $R = 1.08 \cdot 10^{-13}$ cm.

Following ITO *et al.* [6], we get from this figure an « effective angular momentum »,

$$\langle l \rangle = 2.2 .$$

Finally, we observe (Fig. 12) that the disappearance at high energies of the backward « bump », already noticed by O. PICCIONI [7], is confirmed.

[5] M. CHRÉTIEN, J. LEITNER, N. P. SAMIOS, M. SCHWARTZ and J. STEINBERGER: *Phys. Rev.*, **108**, 383 (1957).

[6] D. ITO, S. MINAMI and H. TANAKA: *Nuovo Cimento*, **9**, 208 (1958).

[7] O. PICCIONI: 1958 *Annual International Conference on High Energy Physics at CERN*, p. 65, edited by B. FERRETTI (CERN, Geneva, 1958).

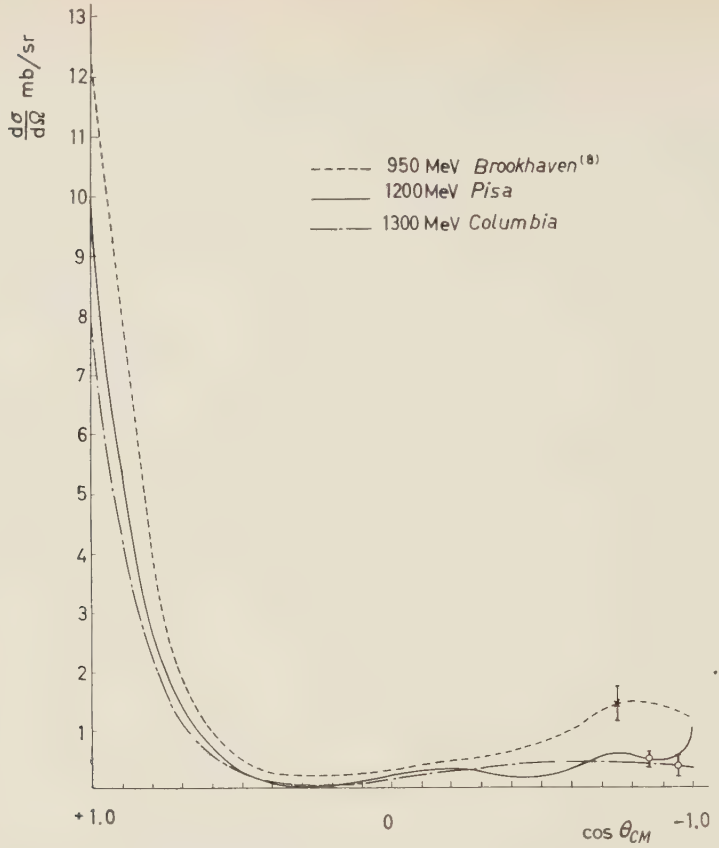

Fig. 12. – Angular distributions at three energies showing the disappearance of the backward « bump ».

3·2. *Total elastic cross-section.* – The number of events, corrected for scanning bias, in the interval $.965 \geqslant \cos\theta_{\text{C.M.}} \geqslant -1$, is 1361 ± 51. From the best fit we find 274 ± 28 events in the interval $1 > \cos\theta_{\text{C.M.}} > .965$.

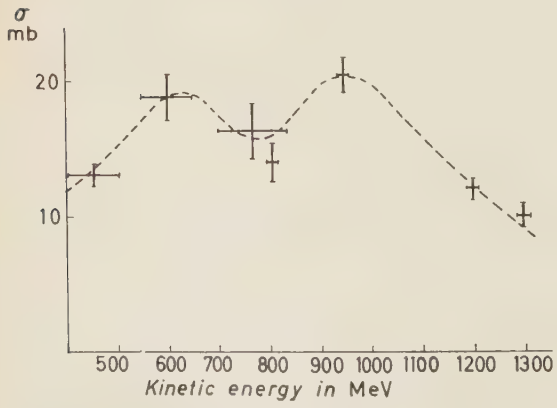

Fig. 13. – Total elastic cross-section *vs.* energy. This plot is reported from R. R. CRITTENDEN *et al.* [4]. We have included the points at 1200 MeV (Pisa) and at 1300 MeV (Columbia [5]).

[8] A. R. ERWIN jr. and J. K. KOPP: *Phys. Rev.*, **109**, 1364 (1958).

To evaluate the total elastic cross-section, from the number of the events we have used the following data:

Total path length	$L = (28.64 \pm 1.37) \cdot 10^5$ cm	(Section 2·6)
Carbon contamination	$C = (1.5 \pm 2.6)\%$	(Section 2·4)
Number of hydrogen atoms per cm^3	$= 4.7 \cdot 10^{22} \pm 2\%$	

We get a total elastic cross-section $\sigma = (12 \pm .8)$ mb. In Fig. 13 our experimental point is plotted together with some points at neighbouring energies.

* * *

We are grateful to the Columbia bubble chamber group, in particular to Prof. J. STEINBERGER, who have kindly sent us the film on which the measurements have been performed. We are indebted to Prof. C. FRANZINETTI for helpful discussion. The financial support from the South African Council for Scientific and Industrial Research to one of us (P.H.S.) and the hospitality received during his stay in Italy are hereby acknowledged. Thanks are due to Dr. A. CHIARINI and the staff of the Centro Calcoli e Servomeccanismi of Bologna University who programmed and carried out the computation for the best fit with the IBM 650.

RIASSUNTO

Si riportano i risultati di una ricerca in camera a bolle a propano sullo scattering da protoni di mesoni di energia cinetica 1200 MeV. La distribuzione angolare dei pioni determinata sperimentalmente è espressa mediante un polinomio del 9° grado in $\cos\theta_{C.M.}$ (dove $\theta_{C.M.}$ è l'angolo di diffusione del π^- nel sistema del baricentro) e il suo valore a 0° è confrontato col valore ottenuto facendo uso delle regole di dispersione e del teorema ottico. La distribuzione angolare ai piccoli angoli, se interpretata come dovuta a scattering di diffrazione, permette di ricavare un valore del raggio ottico del protone. Viene inoltre determinata la sezione d'urto totale per il processo in questione.

Variationen der primären kosmischen α-Teilchen und der Gesamtstrahlung.

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(ricevuto il 16 Agosto 1960)

Summary. — The primary α-particle intensity and the total cosmic radiation was measured over Northern Germany (cut-off for protons 1.46 GeV, eccentric dipol) with a balloon sonde during a strong Forbush effect on July 16, 1959. The vertical α-particle flux extrapolated to 0 g/cm² was found to be $(0.010\,5 \pm 0.000\,7)$ $(\mathrm{cm^2\ s\ sr})^{-1}$ if allowance is made for the fragmentation of heavier nuclei. The extrapolated value of the total radiation amounted to $(0.089\,1 \pm 0.001\,5)$ $(\mathrm{cm^2\ s\ sr})^{-1}$. On November 3, 1959, the measurement of the α-particles showed an extrapolated vertical flux of $(0.012\,4 \pm 0.002\,5)$ particles/$(\mathrm{cm^2\ s\ sr})$ and the total radiation amounted to $(0.104\,3 \pm 0.004\,2)$ particles/$(\mathrm{cm^2\ s\ sr})$ (cut-off for protons 1.72 GeV). A comparison of the radiation values from 16.7.1959, 3.11.1959 and 24.4.1957 proves a variation of the hard component at high altitudes which is larger by the factor 1.9 ± 0.3 than that of neutrons on ground level at Lindau. For the α-particles this variation factor turned out to be 1.5 ± 0.5. The absolute intensities of both components measured during the three flights fit well the modulation scheme of Ehmert [6] which was adjusted to the absolute intensities published by McDonald [7] obtained by different experimental techniques.

Einleitung.

Im Juli 1959 traten starke chromosphärische Eruptionen auf der Sonne auf mit nachfolgenden geomagnetischen Stürmen, welche mit auffallenden Abnahmen der kosmischen Strahlung verbunden waren (s. Abb. 1). Hierbei wurde der tiefste Wert der kosmischen Strahlung im ganzen Sonnenzyklus erreicht.

In dieser Zeit wurde am 16.7.59 mit einem Ballongerät das Ionisationsspektrum der kosmischen Strahlung gemessen. Zum Vergleich fand in der magnetisch weniger gestörten Zeit im November 59 ein weiterer Start statt.

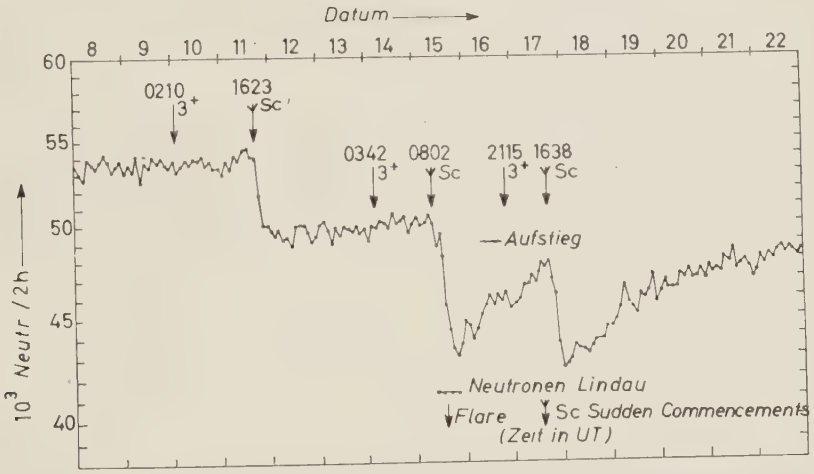

Fig. 1. – Neutronenregistrierung von LINDAU im Juli 1959 korrigiert auf Luftdruck am Erdboden. Die Pfeile markieren die chromosphärischen Eruptionen (flare) und den Beginn der magnetischen Stürme (Sc).

Diese Messungen wurden im Rahmen einer Serie von Aufstiegen durchgeführt, bei denen sowohl die Bestimmung genauer Absolutwerte der totalen primären Strahlung, der Protonen und der Alphateilchen angestrebt wird als auch Einblicke in die Intensitätsschwankungen dieser Komponenten im Vergleich zueinander gewonnen werden sollen. Außerden wird mit dem verwendeten Gerät die Intensität der harten Komponente als Funktion des Luftdruckes gemessen.

Die Unterscheidung zwischen Teilchen verschiedener Ladung erfolgt aufgrund der spezifischen Ionisation, die dem Quadrat der Ladung proportional ist und in unserer Sonde mit Proportionalzählern gemessen wird.

1. – Das Aufstiegsgerät.

Das in Abb. 2 dargestellte Zählrohrteleskop wurde bis auf die neu hinzugefügten seitlichen Geigerzähler an anderer Stelle [1] bereits eingehend beschrieben. Wir können uns daher auf eine kurze Darstellung seiner wesent-

[1] E. WAIBEL: *Mitteilungen aus dem M.P.I. für Aeronomie*, Nr. 1 (Berlin, 1959).

lichen Funktionen anhand des ebenfalls in Abb. 2 eingezeichneten Blockdiagramms der Schaltung beschränken. Das Teleskop soll nur ein Ereignis registrieren, wenn es von einem einzelnen Teilchen durchdrungen wird; dazu ist

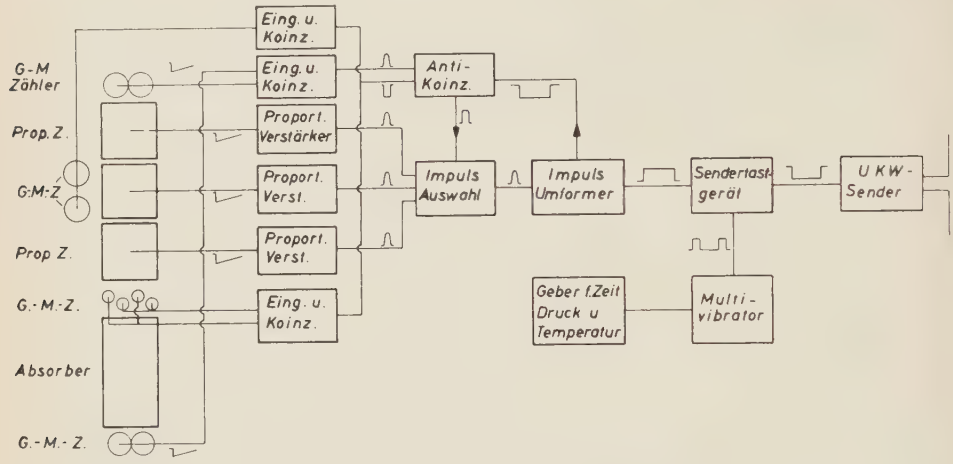

Fig. 2. – Prinzipschaltbild des Aufstiegsgerätes.

eine Koinzidenz von Impulsen der obersten Lage von Geigerzählern, der drei Proportionalzähler und der unterhalb des Absorbers befindlichen Geigerzähler notwendig. Schauer werden dadurch eliminiert, daß nur einer der vier Geigerzähler ansprechen darf, die unmittelbar unterhalb des untersten Proportionalzählers liegen und keiner der seitlich gelagerten Zähler.

Die spezifische Ionisation eines Teilchens wird wegen der starken Unsymmetrie der Ionisationsverteilung nach LANDAU [2] durch elektronische Auswahl des kleinsten Impulses von drei Proportionalzählern gemessen. Die obersten und untersten Geigerzähler begrenzen die Apertur des Teleskops derart, daß sich die in den rechteckigen Proportionalzähler verlaufenden Wege der zugelassenen Teilchen höchstens um 10% unterscheiden.

Der Absorber verhindert eine Registrierung energiearmer einfach geladener Sekundärteilchen, die wegen ihrer relativ hohen Ionisationsverluste Teilchen mit höherer Kernladung vortäuschen würden. Primärteilchen gehen dadurch in größeren Höhen nicht verloren, da die Abschneideenergie des Gerätes genügend unterhalb der geomagnetischen Abschneideenergie für unsere Breiten liegt. Die relativistischen Protonen und Alphateilchen lassen sich auf diese Weise gut unterscheiden. Der Absorber besteht aus 72.0 g/cm² Kupfer und bildet mit den Zählerwänden und 3.2 g/cm² Silicongummi einen Gesamtabsorber von 82.8 g/cm² (äquivalent etwa 10.5 cm Pb). Damit ergibt sich ein Mindest-

[2] L. LANDAU: *Žurn. Èksp. Teoret. Fiz.*, 8, 201 (1944).

wert der kinetischen Energie für Protonen von 320 MeV und für μ-Mesonen von 160 MeV. Die Materieschicht bis zum Erreichen des untersten Proportionalzählers beträgt ca. 1.2 g/cm² Silicongummi und 5.1 g/cm² Messing.

Die Geometriekonstante des Teleskops beträgt für eine isotrope Richtungsverteilung der Strahlung 2.96 cm² sterad und für eine $\cos^2\vartheta$-Abhängigkeit 2.86 cm² sterad (ϑ-Winkel zwischen der Vertikalen und der Einfallsrichtung der Partikel).

Die Impulse der drei Proportionalzähler werden getrennt verstärkt, geeignet geformt und der Schaltung für die Auswahl des kleinsten Impulses zugeführt. Die Impulse der oberen und unteren Geigerzähler werden auf eine Koinzidenzschaltung gegeben, deren Ausgangssignal jedoch in einer Antikoinzidenzstufe unterdrückt wird, wenn gleichzeitig ein Impuls vom Schauerselektor kommt.

Ein erlaubter Impuls von den Geigerzählern erhält dann eine Rechteckform und wird so verstärkt, daß er als vierter Impuls mit stets maximaler Amplitude in die Auswahlschaltung des kleinsten Proportionalimpulses gegeben werden kann. Es handelt sich hierbei also um eine vierfache Auswahl und gleichzeitige Koinzidenz, so daß das Fehlen eines der drei Proportionalimpulse oder des Geigersignals eine Registrierung verhindert. Zur Übertragung per Funk werden diese Impulse variabler Amplituden in Signale variabler Dauer umgeformt. Die Registrierung am Boden erfolgt mit Oszillograph und Filmkamera.

2. – Daten zum Aufstieg vom 16.7.1959.

Der Start des aus 14 Darex 2400 g Neopren-Ballonen bestehenden Gespanns fand am 16.7.1959 um 10.27 in Lindau/Harz geogr. Breite = 51°39′ N, geogr. Länge = 10°07′ E statt. Den während des Fluges registrierten Druck und die entsprechende Höhe gibt Abb. 3 als Funktion der Zeit wieder.

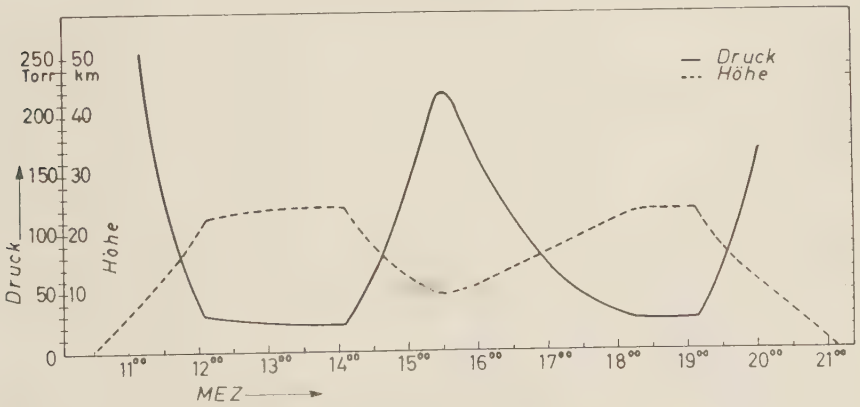

Fig. 3. – Registrierung von Druck und Höhe am 16.7.1959.

Das Gerät war mit einer Ballaststeuerung versehen, die automatisch den Abflußhahn des flüssigen Ballastes öffnete oder schloß, je nach dem, ob der umgebende Luftdruck um einen bestimmten Betrag angestiegen oder abgefallen war. Auf diese Weise sollte versucht werden, das Ballongespann in größtmöglicher Höhe zum Ausschweben zu bringen — nachdem eine geeignete Zahl von Ballonen geplatzt sein würde. Diese Gipfelhöhe ist stets durch die Gleichmäßigkeit des Ballonmaterials vorgegeben. Tatsächlich schwebte das Gespann zum ersten Mal von (12.05÷14.05) MEZ aus. Dann setzte plötzlich eine Abwärtsbewegung ein, offenbar weil ein Ballon geplatzt war. Der Ballastabwurf regelte aber zu dieser Zeit den Auftriebsverlust nicht mehr rasch genug aus, da, wie eine spätere Untersuchung ergab, das intensive Ultraviolettlicht in der großen Höhe Ausfällungen in der Ballastflüssigkeit hervorrief, welche die Ausflußöffnung teilweise verstopften. (Die Flüssigkeitsbehälter waren klarsichtig und sollten eine mäßige Erwärmung der Flüssigkeit durch die Sonnenstrahlung zulassen. Am Erdboden waren diese Störungen ohne sehr starke UV-Einstrahlung nicht feststellbar.) Nach einem Abfangen des Gerätes in 9.6 km Höhe stieg es wieder sehr langsam hoch und schwebte vermutlich allein infolge der Verfestigung der Ballone zum zweiten Mal von 18.20 bis 19.08 MEZ aus. Die während dieser Zeit erreichten Höhen lagen um 21 km. Nach dem Platzen eines weiteren Ballons reichte die noch vorhandene Ballastmenge nicht mehr zur Regelung aus, so daß der endgültige Abstieg erfolgte. Die Landung fand etwa um 21.09 statt.

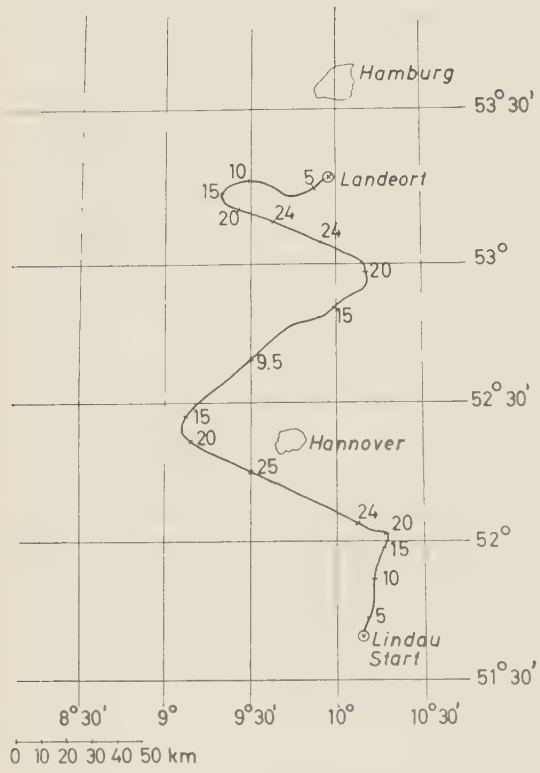

Fig. 4. – Projektion des Flugweges vom 16.7.1959. Die Zahlen an der Kurve geben die Flughöhe in Kilometer an.

Die Projektion der Flugbahn ist in Abb. 4 dargestellt. Bis in Höhen von 19 km herrschte ein Wind aus S bis SW vor, oberhalb aus ESE. Die Bahn

verlief im wesentlichen von Lindau aus in nördlicher Richtung. Die Landung erfolge in der Nähe von Hamburg.

Da sich die geomagnetischen Breiten von Start- und Landungsort um ca. 1.5°, die der Ausschwebestrecken um 1° unterscheiden, müssen die gemessenen Intensitäten auf eine konstante geomagnetische Breite reduziert werden.

3. – Meßergebnisse vom Aufstieg am 16.7.1959.

a) Die harte Komponente. Die Gesamtimpulshäufigkeit der harten Komponente ist als Funktion der atmosphärischen Tiefe in Abb. 5 dargestellt. Es handelt sich hierbei um eine Zusammenfassung der Auf- und Abstiegswerte unter Anbringung der üblichen Korrekturen für Totzeiten und zufällige Koinzidenzen, die jedoch kleiner als 1% bleiben. Man erkennt die Ausbildung eines sehr flachen Maximums. Die statistischen Fehlergrenzen dieser Kurve sind relativ gering, da die Höhenbereiche oberhalb von 10 km viermal durchlaufen wurden. Der Bo-

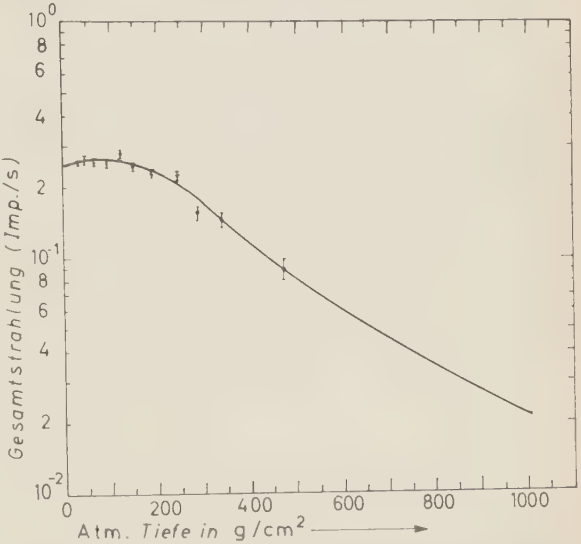

Fig. 5. – Intensität der Gesamtstrahlung als Funktion der atmosphärischen Tiefe (16.7.1959).

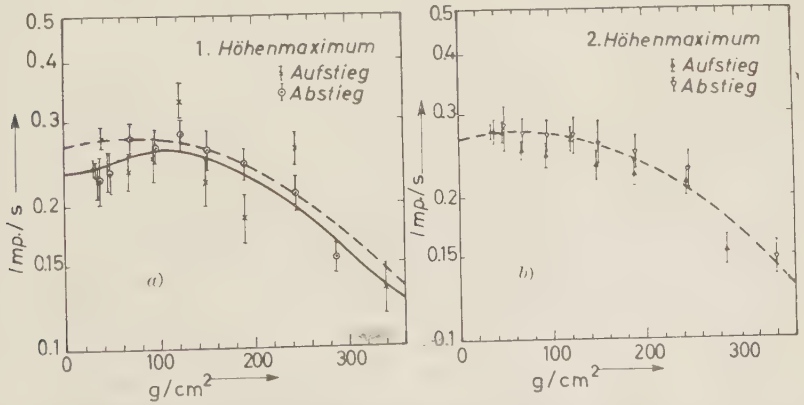

Fig. 6. – Gesamtstrahlung der harten Komponente zum 1. und 2. Höhenmaximum (Fig. 6*a* und 6*b*); zum Vergleich wurde die gestrichelte Linie von Fig. 6*b* auch in 6*a* eingezeichnet. (16.7.1959).

denwert konnte über die Mesonenregistrierungen in Lindau mit guter Genauigkeit ermittelt werden.

Eine detailliertere Registrierung während der einzelnen Zeit- und Höhenphasen zeigt Abb. 6. Die statistischen Fehler sind hierbei für die kürzeren Zeiten größer, aber der Kurvenverlauf zum ersten Höhenmaximum unterscheidet sich deutlich im erwarteten Sinn von dem zum zweiten Ausschwebeteil in etwas höherer geomagnetischer Breite.

b) *Die absoluten Teilchenintensitäten aus vertikaler Richtung.* Die zwischen 20 und 30 Torr gemessenen Häufigkeiten der Gesamtstrahlung werden für die beiden-Ausschwebebereiche und den ganzen Ausftieg nach Korrektur auf Totzeiten und zufällige Koinzidenzen in Tabelle I aufgeführt. Außerdem werden die auf 0 g/cm² extrapolierten Absolutwerte der Gesamtstrahlung und der Protonen mit ihren oberen und unteren Grenzen (wie in [1]) angegeben;

Tabelle I. – *Auf* 0 g/cm² *extrapolierte Strahlungsintensitäten.*

	Obere Grenze Part./(cm² s sr)	Untere Grenze Part./(cm² s sr)	Imp./s bei ca. 35 g/cm²
Gesamtintensität (Mittel)	0.089 1 ± 0.001 5	0.084 2 ± 0.001 5	0.254 ± 0.005
Gesamtintensität 1. Höhenmax.	0.084 0 ± 0.002 0	0.079 4 ± 0.002 0	0.239 ± 0.007
Gesamtintensität 2. Höhenmax.	0.094 7 ± 0.002 8	0.089 5 ± 0.002 7	0.278 ± 0.008
Protonen (Mittel) ohne Berücksichtigung d. Albedo)	0.076 7 ± 0.001 5	0.072 7 ± 0.001 5	
Protonen (Mittel) (mit Berücksichtigung d. Albedo)	0.067	0.063	

letztere zeigen die Berücksichtigung oder Vernachlässigung der Verluste an, die in den oberen Materieschichten des Teleskops (1.2 g/cm² Silicongummi und 5.1 g/cm² Messing) auftreten können. Diese Verluste entstehen durch Auslösung interner Schauer in der oberen Teleskopmaterie und Unterdrückung dieser Ereignisse durch die Schauerantikoinzidenz. Bei Schauern mit sehr enger Bündelung oder solchen geringer Teilchenzahl kann jedoch eine Registrierung erfolgen. Die tatsächlichen Werte liegen also zwischen diesen Grenzen, die sich bei Protonen um einen Faktor von 1.058, bei den Alphateilchen um

1.093 unterscheiden. Eine Albedo (splash albedo) wurde nur bei den Protonen nach ANDERSON [3] berücksichtigt und getrennt angegeben.

Die auf verschwindende Restmaterie extrapolierten α-Intensitäten sind in Tabelle II zusammengestellt. Die Werte wurden auf Totzeiten, zufällige Koinzidenzen und Überlappungen der Ionisationsverteilung korrigiert. Die Extrapolationen wurden nach dem in der früheren Arbeit [1] ausführlich diskutierten Verfahren mit Berücksichtigung der Aufspaltung schwerer Kerne und ohne diese lediglich unter Zugrundelegung einer freien Weglänge in der Luft von 45 g cm^{-2} vorgenommen. Die Absolutintensitäten differieren dadurch nicht unwesentlich; doch ist die sonst notwendige Extrapolation mit Berücksichtigung der Kernaufspaltung durch die stark unterschiedlichen Angaben der Spaltwahrscheinlichkeiten noch mit systematischen Fehlern behaftet. Es wurden hier die Werte von FOWLER, HILLIER and WADDINGTON zugrunde gelegt [4].

TABELLE II. – *Auf 0 g/cm² extrapolierte α-Intensitäten in Teilchen pro* (cm² s sr).

	$A = 45$ g/cm²		mit Korrektur auf Kernaufspaltung	
	obere Grenze	untere Grenze	obere Grenze	untere Grenze
Gesamtaufstieg	0.0134 ± 0.0009	0.0123 ± 0.00085	0.0108 ± 0.0007	0.0099 ± 0.0007
1. Höhenmax.	0.0132 ± 0.0012	0.0121 ± 0.0011	0.0107 ± 0.0009	0.0098 ± 0.0009
2. Höhenmax.	0.0138 ± 0.0015	0.0126 ± 0.0014	0.0110 ± 0.0012	0.0100 ± 0.0011

Die in Tabellen I und II angegebenen Fehlergrenzen enthalten nur die mittleren statistischen Fehler.

4. – Ballonaufstieg vom 3.11.1959.

Zum Vergleich der stark verminderten Strahlungswerte vom 16.7.59 mit der nicht durch einen magnetischen Sturm gestörten Strahlung wurde am 3.11.1959 in Lindau/Harz ein gleichartiges Gerät gestartet. Es flog diesmal in südliche Richtung, so daß die mittlere geomagnetische Breite niedriger war als die von Lindau und die beim Aufstieg am 16.7.1959. Leider kam das Gespann mit 14 Darex 2400 g Neopren-Ballonen nicht zum Ausschweben, da

[3] K. A. ANDERSON: *Suppl. Nuovo Cimento*, **3**, 389 (1957).

[4] P. H. FOWLER, R. R. HILLIER and C. J. WADDINGTON: *Phil. Mag.*, **2**, 293 (1957).

sämtliche Ballone in einer « Kettenreaktion » gleichzeitig in 33.5 km Höhe platzten. Der Abstieg brachte damit praktisch keine Werte mehr; dementsprechend sind die Meßdaten mit relativ großen statistischen Fehlern behaftet. Abb. 7*a* zeigt die Höhenkurve, Abb. 7*b* die registrierte Gesamstrahlung.

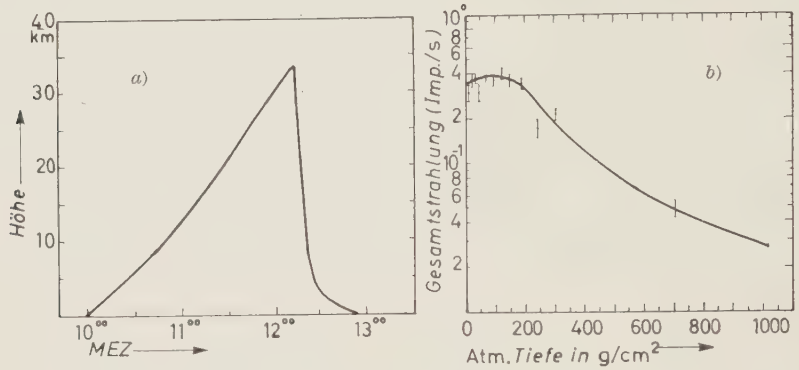

Fig. 7. – *a*) Flughöhe als Funktion der Zeit am 3.11.1959. *b*) Die Gesamtstrahlung am 3.11.1959 in Abhängigkeit von der atmosphärischen Tiefe.

Die auf 0 g/cm² extrapolierten Werte wurden in Tabelle III zusammengefaßt. Die Korrekturen wurden in gleicher Weise wie zuvor angebracht. Die Fehlergrenzen geben die mittleren statistischen Fehler wieder.

Tabelle III. – *Auf* 0 g/cm² *extrapolierte Werte vom* 3.11.1959 *in Teilchen pro* (cm² s sr) *aus vertikaler Richtung.*

	obere Grenze	untere Grenze
Gesamtstrahlung	0.104 3 ± 0.004 2	0.098 6 ± 0.004 0
Protonen (ohne Berücks. d. Albedo)	0.087 9 ± 0.003 6	0.083 0 ± 0.003 4
Protonen (mit Berücks. d. Albedo)	0.075 8	0.071 7
α-Teilchen ohne Korrekt. der Kernaufspaltung ($\Lambda = 45$ g/cm²)	0.015 9 ± 0.003 2	0.014 6 ± 0.002 9
α-Teilchen mit Korrekt. der Kernaufspaltung	0.012 4 ± 0.002 5	0.011 3 ± 0.002 3

Die Geometriekonstante des Teleskops betrug 3.55 cm² sterad für die isotrope Richtungsverteilung und 3.42 cm² sterad für die $\cos^2 \vartheta$ Abhängigkeit.

5. – Vergleich der Strahlungsintensitäten vom 16.7.1959, 3.11.1959 und 24.4.1957.

Wegen der zum Teil großen statistischen Fehler der Messungen am 3.11.1959 kann der Vergleich der stark gestörten Strahlung am 16.7. über Lindau mit einem ungestörten Wert nur von vorläufiger Natur sein. Eine genaue Bestimmung eines derartigen Vergleichswertes ist vorgesehen. Beide Aufstiege wurden in Lindau gestartet, flogen aber in entgegengesetzte Richtungen, so daß die geomagnetische Breite und die Abschneideenergien getrennt zu berechnen waren und die Strahlungsintensitäten zum Vergleich mit den weiter unten ermittelten Reduktionsfaktoren versehen werden mußten. Da sich die Neutronen und Mesonenregistrieranlagen in Lindau befinden, wurde dieser Ort als Bezugspunkt für die Umrechnungen gewählt.

Außerdem ist ein Vergleich mit den Strahlungswerten vom 24.4.1957 über Weissenau interessant, da die Teilchenflüsse nahezu gleich denen vom 16.7.59 sind. Dazu kommt, daß an beiden Tagen dasselbe Maßgerät verwendet wurde und damit eine Unsicherheit in der Geometriekonstanten des Teleskops herausfällt. Es müßte also die Strahlungserniedrigung am 16.7. etwa gleich dem Breiteneffekt sein. Zu bedenken ist allerdings, daß die kosmische Strahlung bei beiden Aufstiegen Forbush-Effekte zeigte. Die Meßergebnisse vom 24.4.57 werden nochmals in Tabelle IV angegeben.

TABELLE IV. – *Auf* 0 g³cm² *extrapolierte Werte vom* 24.4.1957.

	Obere Grenze	Untere Grenze
Gesamtstrahlung	0.096 ± 0.005	0.090 5 ± 0.005
Protonen ohne Berücksichtigung der Albedo	0.083 ± 0.005	0.078 ± 0.005
Protonen mit Berücksichtigung der Albedo	0.072 ± 0.004	0.067 ± 0.004
α-Teilchen ohne Korrektur der Kernaufspaltung ($A = 45$ g/cm²)	0.012 4 ± 0.000 8	0.011 3 ± 0.000 7
α-Teilchen mit Korrektur der Kernaufspaltung	0.010 5 ± 0.000 8	0.009 6 ± 0.000 7

Für die Breitenkorrektur der Strahlungsintensitäten werden die integralen Energie- oder Impulsspektren der Teilchenkomponenten verwendet, bei denen für die Energie bzw. den Impuls die geomagnetischen Abschneidewerte des exzentrischen Dipolfeldes eingesetzt werden, die in dieser Gegend mit denen

TABELLE V. – *Geomagnetische Abschneideenergien in GeV pro N*

	Protonen E_{cp} (in GeV)	α-part. $E_{c\alpha}$ (GeV/Nukl.)	π_p
Lindau	1.62	0.58	2.55
16.7.59 Zusammenfassung	1.46	0.51	2.38
16.7.59 1. Höhenmaximum	1.53	0.54	2.45
16.7.59 2. Höhenmaximum	1.39	0.48	2.29
3.11.59	1.72	0.625	2.68
24.4.57 Weissenau	2.29	0.85	3.31

(*) Umrechnung auf Lindau.

nach der Methode von QUENBY und WEBBER (5) übereinstimmen. Besonders geeignet erscheinen hierfür die von EHMERT (6) für eine elektrische Modulation der Energiespektren berechneten und an die Messungen von MCDONALD (7) angepaßten integralen Spektren für die Protonen und α-Teilchen, da mit ihnen die Sonnenaktivität über die Neutronenregistrierungen am Boden berücksichtigt werden kann. Diese Kurven (s. Abb. 8) wurden unter der Annahme einer Modulation der kosmischen Strahlung durch elektrische Felder berechnet. Die Teilchenhäufigkeiten wurden über dem Impuls/Ruhenergie $\pi = pc/Mc^2$ aufgetragen. Der Kurvenparameter $\mu = zeU/Mc^2$ gibt die Abhängigkeit vom abbremsenden Feld mit der Potentialdifferenz U der Erde gegen den fernen Raum wieder. (z = Teilchenladung, M = Masse, p = Impuls, c = Lichtgeschwindigkeit, e = Elementarladung). Es gilt dann zwischen Protonen und Alphateilchen $\mu_\alpha = \frac{1}{2}\mu_p$.

Die gemessenen absoluten Strahlungsflüsse der α-Teilchen und Protonen der drei Aufstiege wurden in Abb. 8 für die verschiedenen Abschneideimpulse und Gegenspannungen eingetragen. Die Werte von μ wurden aus Ehmerts Diagramm der Weissenauer Neutronenintensitäten nach Umrechnung der Lindauer Registrierungen ermittelt. Die Daten für μ, die Abschneideenergien, Abschneide-

(5) J. J. QUENBY and W. R. WEBBER: *Phil. Mag.*, **4**, 90 (1959).

(6) A. EHMERT: *Proceedings of the Moscow Cosmic Ray Conference*, 1959, **4**, 142 (1960).

(7) F. B. MCDONALD: *Phys. Rev.*, **109**, 1367 (1958).

rrekturen und Strahlungsintensitäten.

	Reduktionsfaktor Protonen	Reduktionsfaktor α-Part.	Neutronen Lindau Werte/2 Stud. f. best. App.	Gesamtstrahlung (30÷160) Torr Imp./s	
				direkt	Red. auf Lindau
	1.000	1.000			
5	0.972	0.972	45 840	(0.253±0.004)	0.246
	0.988		45 650		
	0.955		46 100		
4	1.034	1.028	54 550	(0.300±0.011)	0.311
4	1.188	1.162	51 820 (*)	(0.266±0.007)	0.316

impulse π_p und die aus den integralen Spektren gewonnenen Reduktionsfaktoren für die Breite von Lindau sind in Tabelle V zusammengefaßt. Für die α-Teil-

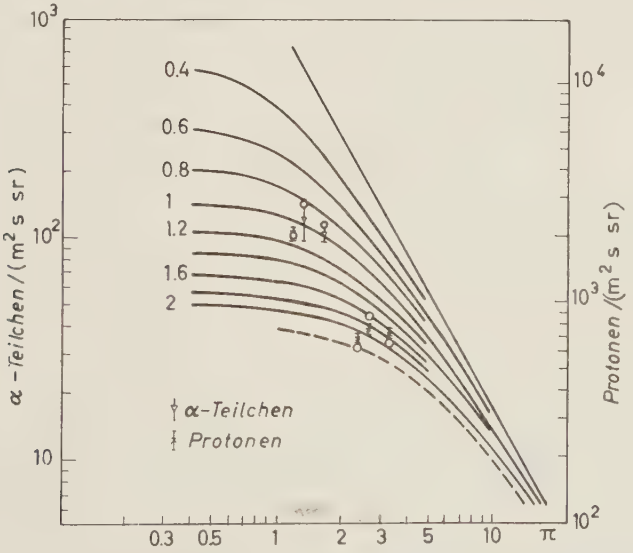

Fig. 8. – Absolute integrale Häufigkeiten von α-Teilchen und Protonen (Albedo nach Messungen von ANDERSON [3]) korrigiert) in Kurvenschar der integralen Impulsspektren nach EHMERT [6]) für verschiedene elektrische Gegenfelder eingezeichnet. Die Kreise geben den nach den Kurven zu erwartenden Wert an.

chen wurden die oberen Grenzwerte der auf Kernaufspaltung korrigierten Werte genommen, für die Protonen die entsprechenden oberen Grenzwerte nach Abzug der Albedo (splash albedo) nach ANDERSON [3]. Die nach den Kurven und μ-Werten aufgrund der Bodenregistrierung zu erwartenden Häufigkeiten wurden als Kreise eingetragen. Man erkennt eine sehr gute Übereinstimmung der Meßwerte mit den Kurven, besonders bei den Alphateilchen aber auch bei den auf Albedo korrigierten Protonenflüssen. Es fällt noch auf, daß die Absolutwerte vom 3.11.1959 gegenüber den anderen allgemein etwas tiefer liegen.

Diese Tatsache wird von besonderer Bedeutung bei der Ermittlung des Verhältnisses der Schwankungen der primären Intensitäten zu denen der Strahlungen dem Erdboden.

Bevor die Schwankungsfaktoren bestimmt werden, sollen einige Intensitätsverhältnisse angegeben werden, und zwar ohne und mit Breitenkorrektur (Tabelle VI). Um beim Vergleich der Gesamtstrahlungen die statistisch bedingten Fehler besonder vom 3.11.1959 herabzudrücken, wurden die Werte der flachen Kurvenbereiche zwischen 30 und 160 Torr zusammengefaßt. Die Ausschweberegistrierungen vom 16.7.1959 wurden hierbei nicht berücksichtigt, da ihren relativ tiefer liegenden Werten ein zu hohes einseitiges Gewicht zukäme. Diese Strahlungswerte sind ebenfalls in Tabelle VI aufgeführt.

TABELLE VI. – *Intensitätsverhältnisse.*

	Gesamtstrahlung bei 0 g/cm²		Gesamtstrahlung 30÷160 Torr		α-Teilchen 0 g/cm² mit Berücks. der Kernafuspaltung	
	direkt	korr. a. Lindau	direkt	korr. a. Lindau	direkt	korr. a. Lindau
J(24.4.57)/J(16.7.59)	1.077	1.316	1.051	1.285	1.00	1.20
J(3.11.59)/J(16.7.59)	1.17	1.25	1.19	1.26	1.18	1.25

In Abb. 9 wurden die Logarithmen der gemittelten Gesamtstrahlung ((30 : 160) Torr), der auf 0 g/cm² extrapolierten Gesamtstrahlung und der Alphateilchenflüsse über den Logarithmen der Neutronenwerte von Lindau aufgetragen.

Der (logarithmische) Schwankungsfaktor beträgt für die Gesamtstrahlungen vom 24.4./16.7. $2.25^{+0.25}_{-0.35}$ und vom 3.11./16.7. 1.50 ± 0.35. Als mittlerer Wert ergibt sich ein Faktor von 1.90 ± 0.30 der mit dem aus den Kurven von Ehmert sich ergebenden übereinstimmt. Wegen der wesentlich größeren Fehler-

grenzen bei den α-Teilchen ist der Schwankungsfaktor bei diesen mit entsprechend großen Fehlern behaftet; er ergibt sich zu 1.5 ± 0.5 und liegt damit scheinbar niedriger als der der Gesamtstrahlung bzw. der Protonen. Ein Faktor von 1.9 liegt jedoch noch innerhalb der Fehlergrenzen.

Von der Differenz der Gesamtstrahlung zwischen dem 1. und 2. Ausschwebebereich in Höhe von *a*) 12% für die extrapolierten Werte bzw. *b*) 15% für die Meßbereiche der Gipfelhöhen lassen sich mit Hilfe der Kurven in Abb. 6 ca. 3.5% aus der Änderung des geomagnetischen Abschneideimpulses und weitere 3% aufgrund der Neutronenänderung am Boden und des ermittelten Schwankungsfaktors erklären. Es bleiben demnach noch für *a*) 5.5% für *b*) ca. 8.5%, denen die statistischen Fehler von zusammen 5.4% gegenüberstehen. In den zusammengefaßten Bereichen zwischen 30 und 160 Torr stimmen die Werte innerhalb der Meßgenauigkeit überein. Die α-Teilchenflüsse unterscheiden sich nur um 2.6%. Wenn man auch aufgrund der Kurvenformen für die harte Komponente (s. Abb. 6*a* und 6*b*) oberhalb von 60 g/cm² Luft eine energieschwache Zusatzstrahlung nicht ausschließen kann, so läßt sich diese Differenz doch im wesentlichen durch die Änderung der geomagnetischen Breite, die geringe auch am Boden erkennbare Strahlungsschwankung und die statistischen Fehler erklären. Die Kurve für die harte Komponente zum 2. Maximum wurde des besseren Vergleichs wegen sowohl in Abb. 6*a* als auch in 6*b* eingezeichnet.

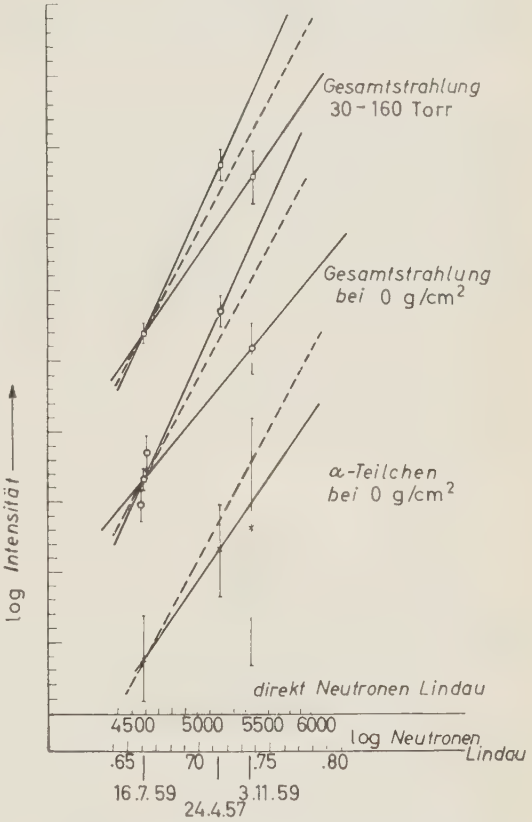

Fig. 9. – Intensitäten der primären Gesamtstrahlung der Alphateilchen und der harten Komponente in größeren Höhen aufgetragen über der Neutronenintensität am Boden in Lindau. Skalen logarithmisch zur Ermittlung der Schwankungsfaktoren.

Trotz der sehr unterschiedlichen Methodik bei den Messungen von McDonald (7) mit einem Szintillations- und einem Čerenkovzähler und unseren mit drei Proportionalzählern und einem Absorber wurde innerhalb der Meßgenauig-

keit durch Einbeziehung der Variationen mit Hilfe der Bodenregistrierungen eine sehr gute Übereinstimmung der Absolutwerte erzielt. Das besagt aber gerade, daß der Modulationsmechanismus bei magnetischen Stürmen und im Rahmen des Sonnenzyklus in gleicher Weise erfolgt (6).

* * *

Dem Direktor des Institutes für Stratosphären-Physik im Max-Planck-Institut für Aeronomie, Herrn Prof. Dr. J. BARTELS, danke ich für seine freundliche Förderung dieser Arbeit. Herrn Prof. Dr. A. EHMERT und Herrn Dr. G. PFOTZER danke ich für förderliche Diskussionen. Die Deutsche Forschungsgemeinschaft hat auf Antrag von Herrn Prof. A. EHMERT die vorliegende Arbeit unterstützt, wofür an dieser Stelle herzlich gedankt wird.

RIASSUNTO (*)

Il 16 Luglio 1959 si sono misurate con un pallone sonda l'intensità delle particelle α primarie e la radiazione cosmica totale sulla Germania Settentrionale (cut-off dei protoni 1.46 GeV, dipolo eccentrico) durante un forte effetto Forbush. Si è trovato che il flusso verticale di particelle α estrapolato a 0 g/cm^2 è (0.0105 ± 0.0007) (cm^2 s sr)$^{-1}$ se si tiene conto della frammentazione dei nuclei più pesanti. Il valore estrapolato della radiazione totale ammontava a (0.0891 ± 0.0015) (cm^2 s sr)$^{-1}$. Il 3 Novembre 1959 la misura delle particelle α mostrava un flusso verticale estrapolato di (0.0124 ± 0.0025) particelle/(cm^2 s sr) e la radiazione totale ammontava a (0.1043 ± 0.0042) particelle/(cm^2 s sr) (cut-off dei protoni 1.72 GeV). Un confronto fra i valori della radiazione dei giorni 16.7.1959, 3.11.1959 e 24.4.1957 dimostra una variazione della componente dura a grandi altitudini maggiore per un fattore 1.9 ± 0.3 di quella dei neutroni al livello del suolo a Lindau. Per le particelle α questo fattore di variazione risultava di 1.5 ± 0.5. Le intensità assolute delle due componenti misurate durante i tre voli si inseriscono bene nello schema di modulazione di Ehmert (6) che fu adattato alle intensità assolute pubblicate da MCDONALD (7), ottenute con tecniche sperimentali diverse.

(*) *Traduzione a cura della Redazione.*

Coulomb Photoproduction of π^0 at High Energy and π^0 Lifetime.

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(ricevuto il 27 Settembre 1960)

Summary. — A discussion is presented of the possibility of determining the lifetime of the π^0 using the photoproduction of π^0 in a Coulomb field as suggested by Primakoff. It is first shown how the competition with the coherent nuclear production makes it necessary to use a rather high photon energy. Even at 1 GeV, however, the Coulomb photoproduction dominates over the coherent nuclear one only in a very narrow cone around the forward direction; this cone has an angle of $\sim 3°$ for $\tau = 5 \cdot 10^{-17}$ s and only 2° for $\tau = 5 \cdot 10^{-16}$ s. Assuming that the difficult technical problems connected with these small angles may be solved, it is still necessary to prove that, at these small angles the incoherent production is not dominant. Indeed, though the incoherent production is certainly smaller than the coherent nuclear one by a factor of A (=mass number), the incoherent cross section does not have the factor $\sin^2\theta$ which affects both the Coulomb and the nuclear coherent cross sections. A calculation of the incoherent photoproduction of π^0 at small angles is therefore made using a multipole expansion and sum rules. It is shown that, for lead, the incoherent cross section at small angles is at most of the same order of magnitude of the nuclear coherent one so that it does not dominate the picture. For light nuclei the competition between incoherent and coherent nuclear production at small angles is more unfavourable and depends rather critically on the properties of the individual nucleus chosen. It is concluded that heavy nuclei are the most suited for this experiment and that, at 1 GeV, only lifetimes shorter than some units times 10^{-16} may possibly be measured by this method.

1. – Coulomb photoproduction of π^0 (*).

The purpose of this note is to amplify some remarks by one of us (1) concerning Primakoff's suggestion (2) to determine the lifetime of the π^0. As is well known, Primakoff's method is based on measuring the production of π^0 by photons in the Coulomb field of a nucleus. This process takes place through the same phenomenological interaction:

$$H' = \frac{\eta}{\mu}\int \Phi(\boldsymbol{x})\, \boldsymbol{E}(\boldsymbol{x})\cdot \boldsymbol{H}(\boldsymbol{x})\, \mathrm{d}^3x\,, \tag{1}$$

which governs the decay of the π^0. Through the study of the photoproduction of π^0 in the Coulomb field of a nucleus one can in principle determine the constant η and therefore the lifetime τ of the π^0

$$\tau^{-1} = \frac{\eta^2\mu}{64\pi}\,. \tag{2}$$

Indeed (compare ref. (1)) the differential cross-section for photoproduction of a π^0 in the Coulomb field of a nucleus Z is

$$\mathrm{d}\sigma_c(\theta) = \frac{8}{137}\, Z^2\, \frac{\tau^{-1}}{\mu^3}\, \frac{v^3 \sin^2\theta F^2(|\boldsymbol{k}-\boldsymbol{q}|)}{[1+v^2-2v\cos\theta]^2}\, \mathrm{d}\Omega\,. \tag{3}$$

Here $v = q/\omega_q = q/k$ is the velocity of the produced π^0 ($\omega_q = k$ since we neglect the recoil energy of the nucleus); $F(|\boldsymbol{k}-\boldsymbol{q}|) = \int \varrho(\boldsymbol{x}) \exp[i(\boldsymbol{k}-\boldsymbol{q})]\cdot\boldsymbol{x}\ \mathrm{d}\boldsymbol{x}$ is the form factor of the nucleus and θ is the production angle of the π^0; $\int \varrho(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} = 1$. (**)

The general behaviour of the angular distribution (3) is as follows: the

(*) *Notation.* - The following general notation is used: μ mass of pion, M mass of nucleon, $k(\boldsymbol{k})$ energy (momentum) of the photon, $\omega_q(\boldsymbol{q})$ energy (momentum) of the pion; units in which $\hbar = c = 1$ have been used.

(1) G. MORPURGO: *Suppl. Nuovo Cimento*, **16**, 17 (1960). Paper presented at the 1959 meeting of the Italian Physical Society.

(2) H. PRIMAKOFF: *Phys. Rev.*, **81**, 899 (1951). Only the total cross section for Coulomb production is given in this paper. A numerical error has been corrected in ref. (1) and also, independently, in ref. (5) below.

(**) *Note added in proof.* - Formula (3) is of course valid in first order perturbation theory; how good this is for large Z should be investigated.

(3) Of course the production of π^0 in electron-electron or electron-positron collisions would offer a much more clean way to determine the lifetime of the π^0. Intersecting

cross-section per unit solid angle vanishes in the forward direction; it rises to a very steep maximum at a very small value of θ, given by $\cos\theta^* = 2r_0(1-r)^2$ that is $\theta^* \simeq \mu^2/2k^2$; it then decreases very rapidly due both to the denominator and the form factor. It has also been pointed out in [1] that the order of magnitude of the cross-section, as given by (3), looks very reasonable for an experiment; but that one point was still obscure for a clear interpretation of the experimental data: the competition of the Coulomb production, as given by (3), and the coherent nuclear production of π^0. In this note we shall: *a*) give an estimate of this competition; *b*) evaluate the forward incoherent nuclear production of π^0. Why this last point is important will be explained later.

2. – Coherent nuclear production of π^0.

This is the usual production of π^0 in the nucleus, with the nucleus remaining in the initial state. The amplitude for this process of course interferes with that of the Coulomb production discussed above so that the total coherent cross-section for production of π^0 has an expression of the form

$$\mathrm{d}\sigma_T = \mathrm{d}\sigma_C + \mathrm{d}\sigma_N \pm 2\sqrt{\mathrm{d}\sigma_C \cdot \mathrm{d}\sigma_N}\,,$$

where $\mathrm{d}\sigma_N$ is the pure nuclear coherent cross-section and the plus or minus sign before the last term is a simplification for a possible phase factor.

Since only an order of magnitude can be estimated for $\mathrm{d}\sigma_N$, it is necessary, in order to be able to obtain by this method the lifetime of the π^0, that $\mathrm{d}\sigma_C \gg \mathrm{d}\sigma_N$ [3].

As will be clear from what follows the expression for $\mathrm{d}\sigma_N$ in a nucleus with spin zero is, at least at small angles,

$$\mathrm{d}\sigma_N = CA^2 \sin^2\theta\, F^2(|\boldsymbol{k}-\boldsymbol{q}|)\,\mathrm{d}\Omega\,. \tag{4}$$

beams are needed for this; but it is doubtful whether one can obtain enough intensity. For instance the differential cross section for the process $e^+ + e^- \to \pi^0 + \gamma$ (the e^+ and e^- having opposite momenta p_0) can be easily calculated; the result is, neglecting some unimportant terms,

$$\mathrm{d}\sigma = \frac{1}{\mu^2}\,\frac{\eta^2}{128\pi}\,\frac{1}{137}\left(\frac{k}{E_0}\right)^3(1+\cos^2\theta)\,\mathrm{d}\Omega\,.$$

The total cross section is

$$\sigma = \frac{1}{\mu^2}\,\frac{\eta^2}{48}\,\frac{1}{137}\left(\frac{k}{E_0}\right)^3.$$

In the above formulas E_0 is the energy of each of the electrons, k the energy of the photon, η the same parameter which appears in (1). If the energy is high enough that the condition $(\mu/2E_0)^2 \ll 1$ is satisfied the total cross section is $\sim 0.3\cdot 10^{-33}$ cm^2 for a lifetime value of 10^{-17} s.

Here the reason for the $\sin^2\theta$ factor is simply the conservation of the z component of the angular momentum: the photon has a spin but not the π^0 nor the nucleus, so that the amplitude must vanish in the forward direction; of course the angular dependence at larger angles may be different from the simple $\sin^2\theta$ but, as we shall see, we are interested only in the angles near the forward direction. The factor F^2 in (4) is the same factor which appears in (3); this equality is usually probably not exactly true, but it is a very good approximation ([4]). The factor A^2 is there because we are dealing with a coherent process; of course in this way we neglect the reabsorption of the π^0 once they have been produced. But the experimental results at low energy ([5,6]) where (see below) the Coulomb production is surely unimportant show that a good fit is obtained with an expression of the form (4), the exponent of A being 1.87. Since we do not see a reason for having more reabsorption at the higher energies in which we shall be interested here than at low energy, we have used in (4) essentially the same A dependence found at low energy with the small change $1.87 \to 2$, which implies only a change by a factor of 2 in lead ([7]). It is obvious that all the reabsorption problems are less important for the low A nuclei, so that for the sake of a clear interpretation of the results the light nuclei should be preferable; but as we shall see, they are much less indicated for other reasons. Concerning finally the determination of the constant C in (4) its value depends of course on the energy, since it is directly related to the non spin flip part of the production amplitude on protons and on neutrons. At low energy we have a reasonably accurate expression for this quantity but not at the high energy ($k = 1$ GeV) at which we shall be interested here. At 1 GeV we write:

$$C = \frac{1}{4}\frac{\sigma_H^0}{4\pi} \simeq 0.5 \cdot 10^{-30}\ \mathrm{cm^2/sr}\,, \tag{5}$$

the rough justification for this value being postponed to the next section.

([4]) Even if at high energy the factor F^2 in (3) and in (4) are somewhat different, which is probably true, this is not important for the subsequent discussion, because we are mainly interested in the very small angles. Notice that at low energy the F^2 factors in (3) and (4) are expected to be essentially equal because of the dominance of the $\frac{3}{2}, \frac{3}{2}$ resonance. We also remark, since we are mentioning the form factors, and to give an idea of the orders of magnitude involved, that at 1 GeV and in lead, the coherent nuclear cross section, due to the reduction from the form factor, is again at around 6° as small as it is at ξ.

([5]) G. DAVIDSON: *M.I.T. Thesis* (September 1959).

([6]) R. SCHRACK, S. PENNER and J. LEISS: *Nuovo Cimento*, **16**, 759 (1960).

([7]) For taking this into account one can, if one likes, divide by two all the figures of the subsequent Table I.

The ratio

$$\frac{d\sigma_N}{d\sigma_C} = C\frac{A^2}{Z^2}\frac{137}{8}\frac{\mu^3}{\tau^{-1}}\frac{(1+v^2-2v\cos\theta)^2}{v^3}, \tag{6}$$

can be written in a convenient form expressing τ in units of $\mu^{-1} \simeq 0.5 \cdot 10^{-23}$ s. On writing:

$$\tau = \mu^{-1}\varrho$$

and using the values $A = 208$, $Z = 82$ and the equation (5) for C we obtain:

$$\frac{d\sigma_N}{d\sigma_C} = 2.8 \cdot 10^{-3}\,\varrho\,\frac{(1+v^2-2v\cos\theta)^2}{v^3}. \tag{6'}$$

It is clear that the minimum value of this expression is obtained for $\theta = 0$; its order of magnitude, at $\theta = 0°$ is $1.7 \cdot 10^{-4}(\mu^2/k^2)^4\varrho$. From this equation, and keeping in mind the values of ϱ, as given in the Table I for reasonable values of τ, we see that it is necessary to go at least to $k = 1$ GeV for having a ratio $d\sigma_N/d\sigma_C$ (at $\theta = 0°$) essentially less than 1. Assuming, therefore to choose $k = 1$ GeV, which means $\mu^2/k^2 \equiv \xi = 2 \cdot 10^{-2} \sim 1°$, we examine the angular distribution of (6'); since a small value of (6') is obtained only for small

TABLE I. – *Values of the ratio* $d\sigma_N/d\sigma_C$ *as a function of* θ, *and for particular values of* τ.

θ	$d\sigma_N/d\sigma_C$	$\tau = 5 \cdot 10^{-17}$ [$\varrho = 10^7$]	$\tau = 5 \cdot 10^{-16}$ [$\varrho = 10^8$]	$\tau = 5 \cdot 10^{-15}$ [$\varrho = 10^9$]
0	$2.8 \cdot 10^{-11}\,\varrho$	$2.8 \cdot 10^{-4}$	$2.8 \cdot 10^{-3}$	$2.8 \cdot 10^{-2}$
ξ	$7 \cdot 10^{-10}\,\varrho$	$7 \cdot 10^{-3}$	$7 \cdot 10^{-2}$	$7 \cdot 10^{-1}$
2ξ	$8 \cdot 10^{-9}\,\varrho$	$8 \cdot 10^{-2}$	$8 \cdot 10^{-1}$	—
3ξ	$3.8 \cdot 10^{-8}\,\varrho$	$3.8 \cdot 10^{-1}$	—	—
5ξ	$2.8 \cdot 10^{-7}\,\varrho$	—	—	—

We have not filled those cases in the table where $d\sigma_N/d\sigma_C$ turns out to be larger than 1. The table refers to $k = 1$ GeV. As described in the text $\xi = \mu^2/k^2 = 2 \cdot 10^{-2}$ rad $\sim 1°$, $\varrho = \tau\mu$. A value of the total cross section on hydrogen of $2.5 \cdot 10^{-23}$ cm^2 has been assumed and the values of A and Z refer to lead.

angles we write

$$\left(\frac{\mathrm{d}\sigma_N}{\mathrm{d}\sigma_C}\right)_{\text{small }\theta} \simeq 2.8\cdot 10^{-3}\,\varrho\left(\frac{1}{4}\,\xi^2+\theta^2\right)^2.$$

The values of this expression are reproduced in the Table I for some values of θ and ϱ.

It is apparent from the table that the nuclear coherent process at 1 GeV constitutes a serious problem; if the lifetime is $5\cdot 10^{-17}$ s one has to use only π^0 produced between 0° and 3°. If it is $5\cdot 10^{-16}$ one has to confine only to the interval 0° to 2°. Whether this is possible or not we do not know; but it is essential to recognize that only if one can be sure that π^0 produced at larger angles are not counted in any appreciable amount one has a measure of the lifetime of the π^0 ([8]). Of course the situation might improve if it could be established that the reabsorption of the nuclear-produced π^0 were more important that we have assumed; although one should than estimate also the possible nuclear reabsorption of the Coulomb-produced π^0, which however may well be negligible; or if better values were known of the non spin flip photoproduction cross-section on nucleons at 1 GeV and if a better theoretical prediction of the cross-section for the coherent nuclear process were available.

In this case one might have the hope to *deduce* the Coulomb cross-section, by using the *theoretical* knowledge of the coherent cross-section. However one would have to obtain a smaller upper limit for the incoherent cross-section than the one which we shall produce in the next section.

3. – Forward incoherent nuclear production of π^0.

The previous discussion has shown that at small angles $\mathrm{d}\sigma_C$ is, at sufficiently high energy, larger than $\mathrm{d}\sigma_N$. However both $\mathrm{d}\sigma_C$ and $\mathrm{d}\sigma_N$ are « small » because of the factor $\sin^2\theta$. In these conditions another problem arises: that of computing the small angle incoherent nuclear production of π^0 and showing, possibly, that this process too has a cross-section, $\mathrm{d}\sigma_N^I(\theta)$, small with respect to $\mathrm{d}\sigma_C(\theta)$. Of course in principle incoherent reactions may be experimentally

([8]) If instead of requiring $\mathrm{d}\sigma_N \ll \mathrm{d}\sigma_C$ one only requires

$$\int_0^\lambda \mathrm{d}\sigma_N \simeq \int_0^\lambda \mathrm{d}\sigma_C\,.$$

the « optimistic » angles λ obtained are a little larger than those obtained from the condition $\mathrm{d}\sigma_N \ll \mathrm{d}\sigma_C$.

distinguished from the coherent process in which we are interested; but in practice this is almost impossible.

Since we are dealing with an incoherent process $d\sigma_N^I(\theta)$ may be expected to be proportional to A and not to A^2,; but, on the other hand, $d\sigma_N^I(\theta)$ does not contain the small $\sin^2\theta$ factor which appears both in $d\sigma_C$ and in $d\sigma_N$. Since at 1 GeV the angle θ^* at which $d\sigma_C$ has its maximum is only 10^{-2} rad, and therefore $\sin^2\theta^* = 10^{-4}$, it is by no means clear that $d\sigma_N^I(\theta)$ is smaller than $d\sigma_C(\theta)$ at those small angles around θ^* at which (compare the conclusion of the last section) $d\sigma_C(\theta)$ is larger than $d\sigma_N(\theta)$. It is therefore necessary to obtain an estimate of $d\sigma_N^I(\theta)$.

We shall for this use the impulse approximation and write the transition operator as $T = \sum_i T_i$, where [9]

$$T_i = [(\boldsymbol{K}_i \cdot \boldsymbol{\sigma}_i + L_i) + (\boldsymbol{M}_i \cdot \boldsymbol{\sigma}_i + N_i)\tau_{3i}] \exp[i(\boldsymbol{k} - \boldsymbol{q})] \cdot \boldsymbol{r}_i . \tag{7}$$

We now make the usual assumption [9] that $\boldsymbol{K}_i$, L_i, $\boldsymbol{M}_i$, N_i do not depend on the nucleon momentum and are therefore independent of the nucleon index i [10].

[9] M. Lax and H. Feschbach: *Phys. Rev.*, **81**, 169 (1951); G. Morpurgo: *Nuovo Cimento*, **8**, 552 (1951); Y. Yamaguchi: *Progr. Theor. Phys.*, **13**, 459 (1955). In this paper a treatment of the coherent photoproduction in He is given to interpret the original experiments of Goldwasser *et al.* (*Phys. Rev.*, **95**, 1692 (1954)).

[10] The cross section (4) for the coherent process discussed in the previous section is easily obtained from the expression (7) given for T (compare Yamaguchi, ref. [9]), noting however that some assumptions in this paper are no more valid at the high energy considered here).

$$d\sigma_N/d\Omega = 2\pi\varrho_F |\langle 0|T|0\rangle|^2 ,$$

where

$$\varrho_F = \left(\frac{q\omega_q}{(2\pi)^3}\right)_{\omega_q = k} .$$

For the evaluation of $\langle 0\ T\ 0\rangle$ we assume that the nucleus has zero total angular momentum; then only the L and N terms in (7) contribute, but the contribution of the N term will be neglected because it is proportional to $(N - Z)/2$.

These terms, as shown in general for instance by Chew *et al.* (*Phys. Rev.*, **106**, 1345 (1958)), are proportional to $(\boldsymbol{k} \wedge \boldsymbol{q}) \cdot \boldsymbol{e}_k$, that is vanish in the forward direction. If we write

$$L_0^2 = (\overline{L^2}/\sin^2\theta)_{\theta=0} , \qquad N_0^2 = (\overline{N^2}/\sin^2\theta)_{\theta=0} ,$$

where the bar over $\overline{L^2}$ and $\overline{N^2}$ refers to an average over the polarization of the photon, we immediately have the formula (4) of the previous section with the constant C given by

$$C = 2\pi\varrho_F L_0^2 .$$

In order to discuss the incoherent production we have to deal with matrix elements of the kind $\langle 0|T|n\rangle$, where $|0\rangle$ is the ground state of the nucleus and $|n\rangle$ some excited state; since we are interested in the very small angles, we can calculate everything in the forward direction: the variation in the incoherent cross-section from the angle 0° to an angle of the order ξ may be neglected.

In the forward direction a great simplification occurs: one can expand the exponential in (7) and confine to the first two terms of the expansion

$$\exp\left[i(\boldsymbol{k}-\boldsymbol{q})\cdot\boldsymbol{r}_i\right] = 1 + i(\boldsymbol{k}-\boldsymbol{q})_z z_i + \dots , \tag{9}$$

where the z axis has been chosen along the forward direction. The legitimacy of this expansion at high energy is due to two different but related circumstances: 1) As we shall see in a moment in the forward direction $|\boldsymbol{k}-\boldsymbol{q}|R$ (where R is the radius of the nucleus) is at most 1.5 for the heaviest nuclei (at the energy of 1 GeV). 2) The matrix elements of the succeeding powers in the expansion (9) have decreasing order of magnitude due to the non relativistic character of the motion of the nucleons in the nucleus.

On the other hand it is necessary, in the expansion (9), to keep the first two terms because *a*) the first term gives rise, as we shall see, to an anomalously small matrix element; *b*) having different parity they contribute to different classes of transitions.

We have (for $k \gg \mu$)

$$(\boldsymbol{k}-\boldsymbol{q})_z \simeq k - \omega_q + \frac{\mu^2}{2\omega_q} \simeq E_n - E_0 + \frac{\mu^2}{2k} , \tag{10}$$

where $E_n - E_0$ is the energy difference of the nuclear system before and after the production of the π^0. To evaluate $(\boldsymbol{k}-\boldsymbol{q})_z R$, we must first obtain an estimate of $E_n - E_0$. We shall now show that $E_n - E_0$ cannot be, when the π^0

Assuming *in the absence of more information* that at 1 GeV

$$I_0^2 \simeq N_0^2 \simeq \overline{K^2} \simeq \overline{M^2} , \tag{8}$$

and therefore these terms contribute about equally to the total cross-section in Hydrogen, we may write

$$2\pi\varrho_P I_0^2 \simeq \frac{1}{4}\frac{\sigma_{\mathrm{H}}^0}{4\pi} ,$$

and three similar equations with N_0^2, $\overline{K^2}$, $\overline{M^2}$; here σ_{H}^0 is the total cross section for photoproduction in H. If we take, at 1 GeV, the value $\sigma_{\mathrm{H}}^0 = 2.5 \cdot 10^{-29}$ cm² as indicated by the results of Berkelman and Waggoner (*Phys. Rev.*, **117**, 1364 (1960)) we obtain for C the estimate given in the eq. (5).

is produced in the forward direction, much larger than ~ 25 MeV and that even this value is not likely to be frequent. To establish the above conclusion we first remark that the conservation of energy and momentum, by themselves, do not imply, of course, essential restrictions on the energies of the various parts of the system, since we are dealing with a many body problem; therefore some model of the collision is required, but, as we shall show, two rather extreme models lead essentially to the same conclusion. In model *a* we suppose that the nucleus is excited, during the production process, from the ground state to some bound or virtual excited state, and, collectively carries away the recoil momentum. In this case, since the recoil energy of the nucleus can be neglected, it is clear that $E_n - E_0$ is simply the excitation energy of the nucleus, which will be at most of the order of 20 MeV for the light nuclei or 10 MeV for the heavier ones. Taking into account that, if $k = 1$ GeV, $\mu^2/2k \sim 10$ MeV we have in this case $(\boldsymbol{k} - \boldsymbol{q})_z R \leqslant 1$ for all A.

In model *b* we assume that the nucleus behaves as a Fermi gas. In this case the validity of the above conclusion is due to the relatively small value of the Fermi momentum with respect to the momentum of the incident photon. In this model the process is pictured as taking place in a collision with an individual nucleon of the nucleus, which behaves before and after the collision as a free nucleon of momentum respectively $\boldsymbol{p}_0$ and $\boldsymbol{p}$; it is therefore qualitatively clear that a large momentum transfer in the forward direction is not possible if $\boldsymbol{p}_0$ is small as compared to $\boldsymbol{k}$. More quantitatively it is possible to show that the maximum value of $(\boldsymbol{k} - \boldsymbol{q})_z$ is obtained when $\boldsymbol{p}_0$ is orthogonal to $\boldsymbol{k}$; the conservation of energy and momentum then leads to the conclusion that

$$E_n - E_0 = \frac{p_0^2}{2M} + \varepsilon ,$$

where $p_0^2/2M$ is the energy corresponding to the momentum p_0 and ε is the separation energy of a nucleon from the nucleus (assuming the residual nucleus to be left in the ground state). We therefore conclude in this case:

$$(\boldsymbol{k} - \boldsymbol{q})_z \leqslant \frac{p_0^2}{2M} + \varepsilon + \frac{\mu^2}{2k} . \tag{11}$$

Assuming that $p_0^2/2M \sim 20$ MeV, $\varepsilon \sim 8$ MeV and $\mu^2/2k \sim 10$ MeV the equation (10) implies $(\boldsymbol{k} - \boldsymbol{q})_z R \sim 1.5$ for the heavier nuclei ($A = 200$) and of course less for the lighter ones.

We may therefore conclude this discussion by stating that the situation in this problem, as far as the multipole expansion is concerned is similar to that which exist for the photonuclear absorption at an energy of ~ 40 MeV; there all the evidence indicates that such an expansion is acceptable; the

same conclusion can be drawn here; and can be justified supplementing the discussion of the value of $(\boldsymbol{k}-\boldsymbol{q})_z R$ which we have just concluded, with the observation, already made, that the matrix elements of succeeding powers in the expansion (9) have decreasing order of magnitude being essentially proportional to increasing powers of $\langle V \rangle$, where $\langle V \rangle$ is the average velocity of a nucleon in the nucleus.

To proceed in the calculation of $\mathrm{d}\sigma_N^I(0)$ we put

$$T(0) = \sum_i (\boldsymbol{K} + \boldsymbol{M}\tau_{3i})\cdot\boldsymbol{\sigma}_i[1 + i(\boldsymbol{k}-\boldsymbol{q})_z z_i] , \tag{12}$$

where the L and N terms have been omitted because, as we have already remarked [10] they vanish in the forward direction; we may write

$$\frac{\mathrm{d}\sigma_N^I(0)}{\mathrm{d}\Omega} = 2\pi \int \frac{q^2}{(2\pi)^3} \frac{\mathrm{d}q}{\mathrm{d}\omega_q} \sum_n |\langle 0|T(0)|n\rangle|^2 \delta(k + E_0 - E_n - \omega_q)\,\mathrm{d}\omega_q = \tag{13}$$

$$= \sum_n \left[\frac{q\omega_q}{(2\pi)^2} |\langle 0|T(0)|n\rangle|^2\right]_{\omega_q = k-(E_n - E_0)} .$$

In (13) the sum over n is obviously limited to all those states for which $E_n - E_0 < k - \mu$, but it can be safely extended to all the states; the small error which one thus makes is in the direction of increasing the computed cross-section. Another approximation which only negligibly increases the computed cross-section is that of calculating the phase space factor putting $\omega_q = k$; this is permissible since we have already established that, for the great majority of the collisions giving rise to forward π^0, we have $E_n - E_0 \ll k$; also we shall approximate ω_q with k in $\boldsymbol{K}$, $\boldsymbol{M}$ in (13).

Therefore we may rewrite the equation (13) as

$$\mathrm{d}\sigma_N^I(0) = 2\pi \varrho_F \sum_n |\langle 0|T(0)|n\rangle|^2 \mathrm{d}\Omega . \tag{14}$$

where ϱ_F is now the same phase space factor which appears in the coherent process (compare the footnote [10]).

We now put:

$$T_1(0) = \sum_i (\boldsymbol{K} + \boldsymbol{M}\tau_{3i})\cdot\boldsymbol{\sigma}_i$$

and, using (10)

$$T_2(0) = i \sum_i (\boldsymbol{K} - \boldsymbol{M}\tau_{3i})\cdot\boldsymbol{\sigma}_i \left[E_n - E_0 - \frac{\mu^2}{2k}\right] z_i .$$

Since, due to parity, $T_1(0)$ and $T_2(0)$ contribute to transitions into different states in (14) we have

$$\mathrm{d}\sigma_N^I(0) = 2\pi\varrho_F \sum_n \{|\langle 0|T_1(0)|n\rangle|^2 + |\langle 0|T_2(0)|n\rangle|^2\}\,\mathrm{d}\Omega\,. \tag{15}$$

We begin by estimating the first sum $\sum_n |\langle 0|T_1(0)|n\rangle|^2$. It is

$$\sum_n |\langle 0|T_1(0)|n\rangle|^2 = \langle 0|\Big|\sum_i (\boldsymbol{K} + \boldsymbol{M}\tau_{3i})\cdot\boldsymbol{\sigma}_i\Big|^2|0\rangle = \\ = \tfrac{1}{3}|\boldsymbol{K}|^2\langle 0|\Big|\sum_i \boldsymbol{\sigma}_i\Big|^2|0\rangle + \tfrac{1}{3}|\boldsymbol{M}|^2\langle 0|\Big|\sum \boldsymbol{\sigma}_i\tau_{3i}\Big|^2|0\rangle\,,$$

where a ground state spin zero has been assumed. We finally have

$$\sum_n |\langle 0|T_1(0)|n\rangle|^2 = \boldsymbol{K}^2\,\overline{S_z^2} + \boldsymbol{M}^2\,\overline{Y_z^2}\,,$$

where

$$\overline{S_z^2} = \langle 0|\Big|\sum_i \sigma_{iz}\Big|^2|0\rangle\,, \qquad \overline{Y_z^2} = \langle 0|\Big|\sum_i \sigma_{iz}\tau_{3i}\Big|^2|0\rangle\,. \tag{16}$$

The contribution to $\mathrm{d}\sigma_N^I(0)$ from $T_1(\sigma)$ is therefore

$$\mathrm{d}\sigma_N^I(0)_1 = 2\pi\varrho_F(\boldsymbol{K}^2\,\overline{S_z^2} + \boldsymbol{M}^2\,\overline{Y_z^2})\,\mathrm{d}\Omega\,. \tag{17}$$

Now $\overline{S_z^2}$ and $\overline{Y_z^2}$ depend of course on the model of the nucleus chosen. If one uses particularly simple models [11] these quantities may even vanish in the ground state of a spin zero nucleus: but this is certainly an oversimplification. However the important point for us here is that, quite generally, due to the saturating property of the spins both $\overline{S_z^2}$ and $\overline{Y_z^2}$ are « small », that is of the order one and not of the order A. With the meaning of the constant C already discussed [10] we can therefore conclude that the order of magnitude of $\mathrm{d}\sigma_N^I(0)_1$ is

$$\mathrm{d}\sigma_N^I(0)_1 \sim 2C\,\mathrm{d}\Omega \tag{18}$$

independent of A.

[11] For instance models described by wave functions in which all the spins are paired to give a resultant zero spin (of course the orbital angular momenta must then also combine to give zero).

We now consider the second sum $\sum_n |\langle 0|T_2(0)|n\rangle|^2$ in (15). We have

$$\sum_n |\langle 0|T_2(0)|n\rangle|^2 = \langle 0| \sum_i (\boldsymbol{K} + \boldsymbol{M}\tau_{3i}) \cdot \left(\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{\sigma}_i z_i + i \frac{\mu^2}{2k} \boldsymbol{\sigma}_i z_i \right)^2 |0\rangle \ .$$

where use has been made of the fact that, for any time independent operator

$$\langle 0| \frac{\mathrm{d}}{\mathrm{d}t} \Omega |n\rangle = i(E_n - E_0) \langle 0|\Omega|n\rangle \ .$$

In the expression (19) the time derivation produces the two terms $\dot{\boldsymbol{\sigma}}_i z_i + \boldsymbol{\sigma}_i \dot{z}_i$ of which the first can be presumably neglected: indeed if we estimate the rate of change of a spin in the nucleus induced by a spin orbit coupling ([12]), the order of magnitude of this rate is determined by the coefficient a of the spin orbit coupling interaction which is of the order of 0.5 MeV for lead and therefore negligible with respect to the term in (19) having as coefficient $\mu^2/2k$.

We therefore consider only the expression

$$\langle 0| \sum_i (\boldsymbol{K} + \boldsymbol{M}\tau_{3i}) \cdot \boldsymbol{\sigma}_i \left(\dot{z}_i + i \frac{\mu^2}{2k} z_i \right)^2 |0\rangle \ . \tag{20}$$

In forming the square of the operator contained in (20) we first evaluate the one particle terms; they give

$$\langle 0| \sum_i |\boldsymbol{K} + \boldsymbol{M}\tau_{3i}|^2 \left(\frac{p_{zi}^2}{M^2} + \frac{\mu^4}{4k^2} z_i^2 \right) |0\rangle \ , \tag{21}$$

where use has been made of the relation $z_i = p_{zi}/M$; a relation which is only approximate due to the exchange forces.

Neglecting the terms linear in τ_{3i}, we obtain for the part of $\mathrm{d}\sigma_N^I(0)_2$ due to the one particle terms the estimate (at $k = 1$ GeV)

$$2CA \, \mathrm{d}\Omega \, (10^{-2} + 10^{-3} A^{\frac{2}{3}}) \ , \tag{22}$$

where we have used

$$\langle 0| \frac{p_{zi}^2}{M^2} |0\rangle = \frac{2}{3M} \langle 0| \frac{p_i^2}{2M} |0\rangle \sim 10^{-2} \ ,$$

([12]) Compare the estimate given in: D. R. INGLIS: *Rev. Mod. Phys.*, **25**, 390 (1953).

and

$$\langle 0|z_i^2|0\rangle = \tfrac{1}{3}\langle 0|r_i^2|0\rangle = \tfrac{1}{5}R^2 \sim \tfrac{1}{5}(\mu^{-1}A^{\frac{1}{3}})^2 .$$

Consider now the two particles terms; an argument similar to that used by LEVINGER and KENT ([13]) in connection with a computation of the mean absorption energy for the nuclear photoeffect shows that such terms give a negative contribution to the expression (20); notice, by the way, that the fact that $\overline{S_z^2}$ and $\overline{Y_z^2}$ in (17) are not of order A, but at most of order unity, is due to a complete or almost complete cancellation of the one particle terms in the expression of $\overline{S_z^2}$ or $\overline{Y_z^2}$ by the two particle terms. This cancellation will not be so complete for the operator in (20), but the point of interest for us is that, in any case, the expression (21) can be regarded as an upper limit to the value of (20). It follows that

$$\mathrm{d}\sigma_N^I(0)_2 \leqslant 2CA(10^{-2} + 10^{-3}A^{\frac{2}{3}})\,\mathrm{d}\Omega . \tag{23}$$

We can therefore conclude, taking into account the equation (18), that

$$\mathrm{d}\sigma_N^I(0) \leqslant 2C(10^{-2}A + 10^{-3}A^{\frac{5}{3}} + 1)\,\mathrm{d}\Omega .$$

If we now consider the ratio of the nuclear incoherent production to the nuclear coherent one, at the angle θ^* and at 1 GeV we obtain

$$\frac{\mathrm{d}\sigma_N^I(\theta^*)}{\mathrm{d}\sigma_N(\theta^*)} \simeq \frac{\mathrm{d}\sigma_N^I(0)}{\mathrm{d}\sigma_N(\theta^*)} \leqslant \frac{2[10^{-2}A + 10^{-3}A^{\frac{2}{3}} + 1]}{A^2\theta^{*2}} \sim \tag{25}$$

$$\simeq \frac{2\cdot 10^4[10^{-2}A + 10^{-3}A^{\frac{2}{3}} + 1]}{A^2} .$$

For $A = 200$ we have $\mathrm{d}\sigma_N^I\theta^*)/\mathrm{d}\sigma_N(\theta^*) < 5$ so that at the angle θ^* the incoherent production in heavy nuclei at 1 GeV is at most 5 times the coherent production.

The same conclusion is not necessarily true for small values of A. From equation (25) we have, for $A = 10$, $\mathrm{d}\sigma_N^I(\theta^*)/\mathrm{d}\sigma_N(\theta^*) \leqslant 2\cdot 10^2$. In such case if the estimate of the expression (17) given by the equation (18) is not too far from truth, the forward incoherent production is a dangerous competitor to the Primakoff production for $\tau \quad 5\cdot 10^{-16}$. However if is necessary, before this last conclusion can be established, to know more about the quantities $\boldsymbol{K}$ and $\boldsymbol{M}$ and to perform, for each individual nucleus an accurate evaluation of $\overline{S_z^2}$ and $\overline{Y_z^2}$.

([13]) J. LEVINGER and D. KENT: *Phys. Rev.*, **95**, 418 (1954).

4. – Conclusion.

We may summarize our results by stating that with much ability [14] one may hope to measure, at 1 GeV, the lifetime of the π^0 by this method only if the lifetime is $< 5 \cdot 10^{-16}$ s. The best nuclei for this purpose are the heavy ones, because in the light ones we cannot be sure, unless more accurate calculations are performed for each individual case, that the incoherent process may not produce big contaminations; indeed the indications are that such incoherent process in the light nuclei may be the main reason of trouble [15]. On the other hand for heavy nuclei we have shown that at those small angles $0 < \theta < 1 \rightarrowtail 3\xi$ at which it is necessary to confine the measurement to exclude a competition from the coherent process, and at which, consequently, both the coherent and the Primakoff process are enormously reduced by the $\sin^2\theta$ factor, the incoherent process is at most of the same order of magnitude, at $\theta = \theta^*$, as the coherent nuclear process.

We finally add that of course one would like to have a direct experimental confirmation of the A^2 or $A^{1.87}$ law at high energy; but that even if the exponent of A in (4) would turn out to be smaller, necessarily not much could be gained, because of the incoherent production.

* * *

The results presented by Dr. TOLLESTRUP at the 1960 Rochester conference and a discussion which one of us (G.M.) had with Dr. TOLLESTRUP have stimulated the study of the incoherent process presented here; we would like to thank Dr. TOLLESTRUP for this. Discussions with D. LEISS and Dr. OSBORNE on the situation at low energy are also gratefully acknowledged. Also we thank Drs. BELLAMY, BELLETTINI and FOÀ (Universities of Florence and Pisa) who, since a few months, have in preparation an experiment on this subject to be made in Frascati; their interest in these questions has been useful. Finally exchange of information with Dr. FERRELL who has been thinking about the same problem is gratefully acknowledged.

[14] Notice that we have not considered in this paper the additional complications due to the brehmsstrahlung spectrum.

[15] Of course another obvious advantage of the heavy nuclei is their larger cross section.

RIASSUNTO

Viene discussa la possibilità di misurare la vita media del π^0 per mezzo della fotoproduzione di π^0 in un campo coulombiano, secondo una proposta di Primakoff. Si dimostra dapprima come la competizione di questo processo con la produzione nucleare coerente renda necessario il ricorso a fotoni di energia abbastanza elevata. Tuttavia, anche a 1 GeV, la fotoproduzione coulombiana predomina su quella nucleare coerente solo in un cono assai stretto attorno alla direzione dei γ incidenti; l'apertura di questo cono è di 3° per $\tau = 5 \cdot 10^{-17}$ s e soltanto di 2° per $\tau = 5 \cdot 10^{-16}$ s. Supponendo che le difficoltà tecniche connesse a misure ad angoli piccoli possano essere superate, rimane ancora da provare che, a questi angoli, non sia predominante la fotoproduzione incoerente. Infatti, per quanto la produzione incoerente sia certamente minore per un fattore A (= numero di massa) di quella nucleare coerente, la sezione d'urto incoerente non contiene il fattore $\sin^2 \theta$ che si trova invece sia nella sezione d'urto coulombiana, sia in quella coerente. Viene perciò effettuato un calcolo per la fotoproduzione incoerente ad angoli piccoli, usando uno sviluppo in serie di multipoli e regole di somma. Si dimostra che, per il piombo, la sezione d'urto incoerente, per angoli piccoli è al più dello stesso ordine di grandezza di quella nucleare coerente, in modo tale cioè da non essere il termine predominante. Per nuclei leggeri la competizione tra produzione nucleare coerente ed incoerente è più dannosa e dipende piuttosto criticamente dalle proprietà del particolare nucleo considerato. Si conclude che i nuclei pesanti sono i più adatti all'esperienza che a 1 GeV questo metodo di misura della vita media è possibile solo se la vita media è minore di qualche unità in 10^{-16} s.

IL NUOVO CIMENTO Vol. XIX, N. 3 1° Febbraio 1961

Notes on the Static Parity Non-Conserving Internucleon Potential.

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(ricevuto il 10 Ottobre 1960)

Summary. — If the weak Lagrangian density $\mathscr{L}$ consists of self-interacting charged currents, the inclusion into these of a term linear in the pion field leads to a static parity non-conserving internucleon potential V, the general order of magnitude of whose effects no longer depends on the quadratic pion terms needed for the conservation of the vector current. If neutral currents also contribute to $\mathscr{L}$ in such a way as to make it an isotopic scalar, the inclusion of linear pion terms has no effect on V, which then remains inappreciable unless the vector current is conserved. The origins of such pion terms are discussed; it emerges incidentally that as a consequence of time reversal invariance, V cannot contain contributions arising from the exchange of a single pion.

1. – Introduction.

The observation that the exchange of leptons entails an internucleon potential is as old as field theory itself. Since the nucleon-lepton interaction is not reflection-invariant, this potential is parity non-conserving (PNC in the following); but being quadratic in the weak coupling constant G it is, from a quantitative point of view, completely negligible.

More recently however it has been suggested that the weak Lagrangian density, $\mathscr{L}$, is of the self-interacting current form (SIC in the following):

$$\mathscr{L} = GJ_\mu J_\mu^* \,. \tag{1.1}$$

Such a form of $\mathscr{L}$ is intrinsically appealing on grounds of elegance and lack of internal arbitrariness; more important, it is a consequence of the picture

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according to which the weak decays proceed as two stage processes via a virtual intermediate boson (1). Along with the terms responsible for the observed decays, $\mathscr{L}$ then necessarily contains interactions which lead to a PNC internucleon potential of first order in G whose exact strength, range, and symmetry properties depend on the details of J_μ. In the following we shall confine ourselves entirely to the static (momentum-independent) part of such a potential, which presumably is dominant, and denote it by $\mathscr{V}$.

The form of $\mathscr{V}$ entailed by the simplest possible expression for J_μ has recently been evaluated to lowest order on the Chew-Low (static nucleon) model by Blin-Stoyle (2), who has also considered different ways of detecting it experimentally (3). He considers only charged currents and allows in them only those terms which are directly necessary if $\mathscr{L}$ is to account for the observations on nuclear β-decay. These include in particular the polar vector bilinear in the pion fields postulated by the conserved vector current hypothesis of Feynman and Gell-Mann (4), so that the strongly interacting nonstrange part of J_μ, which alone is relevant to $\mathscr{V}$, becomes (*)

$$J_\mu = \overline{\psi} i\gamma_\mu(1 + i\gamma_5)\tau_+\psi + i\sqrt{2}\,(\varphi_+\partial_\mu\varphi_0 - \varphi_0\partial_\mu\varphi_+)\,. \tag{1.2}$$

Blin-Stoyle makes the fundamental observation that the effect on $\mathscr{V}$ of the various terms in $\mathscr{L}$ are very different, even though they are all prefaced by the same constant G. Thus the static part of the four nucleon contact interaction is actually parity conserving, and the importance even of the PNC momentum-dependent component is largely if not entirely suppressed by the hard core of the (PC) internucleon potential, which prevents the wavefunction from penetrating into the region where this component can act. With his form of $\mathscr{L}$ it turns out that the effective parts of $\mathscr{V}$ derive entirely from the cross terms between the meson and nucleon currents, so that experiments on PNC effects in strongly interacting systems actually test the SIC and con-

(*) As regards the γ matrices and the Hermiticity of the spinor covariants we adopt the notation and conventions of Jauch and Rohrlich (5). The i-spin matrices are defined by $\tau_\pm = (\tau_x \pm i\tau_y)/2$, $\tau_0 = \tau_z$, so that the actual $\pm$ components of the Pauli vector $\boldsymbol{\tau}$ are $\sqrt{2}\tau_\pm$. By contrast the corresponding components of the i-vector pion field, $\boldsymbol{\varphi}$, are $\varphi_\pm = 2^{-\frac{1}{2}}(\varphi_x \pm i\varphi_y)$. This convention, although making it ackward to keep track of factors of $\sqrt{2}$ while dealing with the operators, appears to be too firmly entrenched to ignore. Note finally that $\varphi_\mp$ emits $\pi^\mp$ and absorbs $\pi^\pm$.

(1) N. Byers and R. E. Peierls: *Nuovo Cimento*, **10**, 520 (1958).
(2) R. J. Blin-Stoyle: *Phys. Rev.*, **118**, 1605 (1960).
(3) R. J. Blin-Stoyle: *Phys. Rev.*, **120**, 181 (1960).
(4) R. P. Feynman and M. Gell-Mann: *Phys. Rev.*, **109**, 193 (1958).
(5) J. M. Jauch and F. Rohrlich: *The Theory of Photons and Electrons* (Cambridge, Mass., 1955).

served vector current hypotheses together; such effects would be effectively absent if either of these hypotheses is incorrect. Within their framework Blin-Stoyle finally estimates the order of magnitude of the ratio of amplitude-dependent PNC effects to their PC counterparts as 10^{-7}. This is roughly one order of magnitude below the upper limit set by experiments to date.

The point of departure of the present paper is the likelihood that a linear (pseudovector) pion term occurs in J_μ. Since the effects on $\mathscr{V}$ of different types of terms in J_μ vary so widely, it is of interest to trace the consequences of this possibility. The ancestry of such terms is discussed in Section 3, after some preliminary remarks in Section 2 on PNC nucleon-pion couplings, from which we deduce incidentally that in the static limit no one-pion-exchange terms can occur in $\mathscr{V}$. In Section 4 we consider to what extent a linear pion term makes it possible to test the SIC form of $\mathscr{L}$ independently of the conserved vector current hypothesis. Such a decoupling of the consequences of the two hypotheses is desirable both on esthetic grounds, and also because the predictions of the second have not as yet received experimental confirmation [6] apart from the non-renormalization of the vector β coupling itself which the hypothesis was originally put forward to explain.

Finally in Section 5 we shall look at the consequences of assuming different transformation properties of $\mathscr{L}$ in i-space, such as follow from the specific proposals of LEE and YANG [7] concerning the intermediate bosons.

The appendix deals briefly with the equivalence transformation which is a necessary though tacit intermediate step if the strong pion-nucleon coupling is to be considered according to the static model. It appears that in contrast to the pionic renormalization problem [8] the leading terms in $\mathscr{V}$ are not qualitatively affected by the transformation.

2. – PNC nucleon-pion coupling and the absence of one-pion-exchange terms.

The object of this section is to reach some general conclusions about PNC nucleon-pion interactions which are linear in the pion field and bilinear and non-derivative in the nucleon field, in order to prepare for the discussion of couplings which arise when a linear pion term is introduced into J_μ. We shall determine which of these couplings are invariant under time-reversal, without prejudicing the question whether they occur directly in the Lagrangian density or are simulated by pionic corrections to a different type of vertex.

[6] H. WEIDENMUELLER: *Phys. Rev. Lett.*, **4**, 299 (1690); cf. also references given in this paper.

[7] T. D. LEE and C. N. YANG: *Phys. Rev.*, **119**, 1410 (1960).

[8] G. BARTON: *Phys. Rev. Lett.*, **2**, 224 (1959).

Conforming to current theory we shall retain only those which are time-reversal invariant, and show that in the static limit for nucleons they cannot give rise to one-pion-exchange contributions to v^c. The absence of such contributions has already been noted by BLIN-STOYLE [2] in the special case considered by him.

Our time reversal is the Wigner type treated in reference [5], under which ψ transforms as follows:

$$\psi \to \psi' = D\psi^c \,, \tag{2.1}$$

where the superscript c denotes the operator which is complex conjugate in Hilbert space, and where D satisfies

$$\gamma_\mu'^{*} = D^{-1}\gamma_\mu D \,, \tag{2.2}$$

the star denoting the complex conjugate matrix, and γ_μ' being given in (2.4) below.

By time-reversal invariance of an operator Q we mean that under this transformation it behaves as follows:

$$Q \to Q' = Q^c \;;$$

by saying that Q changes sign we mean

$$Q \to Q' = -Q^c \,.$$

For brevity in the following we shall omit writing the superscript c on the right-hand side; its presence is always to be kept in mind, especially in conjunction with the factors of i which are needed to make some of the spinor covariants Hermitean.

Under time reversal, the Hermitean covariants transform as follows:

$$\left\{\begin{aligned}
&\overline{\psi}\psi \to \overline{\psi}\psi \,,\\
&\overline{\psi}\gamma_5\psi \to -\overline{\psi}\gamma_5\psi \,,\\
&\overline{\psi}i\gamma_\mu\psi \to -\overline{\psi}i\gamma_\mu'\psi \,,\\
&\overline{\psi}\gamma_5\gamma_\mu\psi \to -\overline{\psi}\gamma_5\gamma_\mu'\psi \,,\\
&\overline{\psi}i\sigma_{\mu\nu}\psi \to -\overline{\psi}i\sigma_{\mu\nu}'\psi \,,
\end{aligned}\right. \tag{2.3}$$

where γ_μ', $\sigma_{\mu\nu}'$ are the regular transforms of γ_μ, $\sigma_{\mu\nu}$; *i.e.*

$$\gamma_0' = -\gamma_0 \,, \qquad \underline{\gamma}' = +\underline{\gamma} \,. \tag{2.4}$$

The transformation property of the pion field $\boldsymbol{\varphi}$ is determined by requiring the conventional PS-PS nucleon-pion coupling $\overline{\psi}\gamma_5\boldsymbol{\tau}\cdot\psi\boldsymbol{\varphi}$ to be invariant. In view of (2.3) this implies (noting that $\tau_\pm$, τ_0 are real)

$$\varphi_0 \to -\varphi_0\,, \qquad \varphi_\pm \to -\varphi_\pm\,. \tag{2.5}$$

The independent non-derivative PNC Hermitean couplings are

$$\left|\begin{array}{l} \mathscr{L}_1 = \overline{\psi}\boldsymbol{\tau}\cdot\boldsymbol{\varphi}\psi\,, \\ \mathscr{L}_2 = \overline{\psi}(\tau_+\varphi_- - \tau_-\varphi_+)\psi\,, \\ \mathscr{L}_3 = i\overline{\psi}(\tau_+\varphi_- - \tau_-\varphi_+)\psi\,, \\ \mathscr{L}_4 = \overline{\psi}\psi\varphi_0\,. \end{array}\right. \tag{2.6}$$

Inspection of (2.3) at once rules out all but $\mathscr{L}$; thus the only surviving candidate is

$$\mathscr{L}_3 = i\overline{\psi}(\tau_+\varphi_- - \tau_-\varphi_+)\psi = \overline{\psi}(\boldsymbol{\tau}\times\boldsymbol{\varphi})_0\psi\,. \tag{2.7}$$

The independent derivative Hermitean couplings are

$$\left|\begin{array}{l} \mathscr{L}_5 = \overline{\psi}i\gamma_\mu\boldsymbol{\tau}\psi\cdot\partial_\mu\boldsymbol{\varphi}\,, \\ \mathscr{L}_6 = \overline{\psi}i\gamma_\mu\,(\tau_+\partial_\mu\varphi_- + \tau_-\partial_\mu\varphi_+)\psi\,, \\ \mathscr{L}_7 = i\overline{\psi}i\gamma_\mu(\tau_+\partial_\mu\varphi_- - \tau_-\partial_\mu\varphi_+)\psi\,, \\ \mathscr{L}_8 = \overline{\psi}i\gamma_\mu\psi\partial_\mu\varphi_0\,. \end{array}\right. \tag{2.8}$$

There are no tensor couplings in (2.8) because we are not allowing derivatives of the spinor fields and because terms of the type $\bar{\psi}i\sigma_{\mu\nu}\psi\partial_\mu\partial_\nu\varphi$ vanish identically, irrespective of how we chose the combinations in i-space. Furthermore, even if spinor derivatives were admitted, the tensor terms all vanish in the limit of static nucleons.

The requirement of time-reversal invariance eliminates only $\mathscr{L}_7$, but the interactions in (2.8) are all vector couplings of a (pseudo) scalar meson, and by virtue of nucleon conservation $\mathscr{L}_8$ is actually a total four-divergence and therefore ignorable.

If only one meson is exchanged between the two nucleons then it carries the entire momentum transfer, say k, and dropping induced terms, which vanish in the static limit, the vertex at which $\mathscr{L}_5$ or $\mathscr{L}_6$ act vanishes by virtue of the equality

$$\overline{u}_f\gamma_\mu k_\mu u_i = u_f\gamma_\mu(p_f - p_i)_\mu\overline{u}_i = 0\,. \tag{2.9}$$

This leaves $\mathscr{L}$, eq. (2.7), as the only possible PNC vertex of the type prescribed by which one-meson-exchange-terms may enter γ. However, $\mathscr{L}'_3$ is the 0 component of a vector in i-space, and we shall see below that none of the fundamental PNC couplings have i-vector constituents (*), whence, although admissible under time reversal, $\mathscr{L}_3$ does not occur in practice.

This completes the demonstration that no one-pion-exchange terms occur in γ, a simplification to which we shall appeal frequently in the following.

3. – Linear pion term in the self-interacting current.

The axial vector part of J_μ as given by (1.2), *i.e.*

$$A_\mu = -\overline{\psi}\gamma_\mu\gamma_5\tau_+\psi\,, \tag{3.1}$$

is just that suggested by the Universal Fermi Interaction picture of the weak decays. Since there is as yet no method for performing entirely credible dynamic calculations of pionic effects on the final manifestations of the weak interactions, we have no way of deciding rigorously whether or not (3.1) can account for all the relevant data, chief among which are the charged pion lifetime and the observed renormalization of the axial vector coupling constant in nuclear β-decay (its unrenormalized or bare value being taken, by appeal to the Universal theory, from μ-decay)[9,10].

However, even allowing for this general uncertainty, attempts to derive the pion lifetime on the basis of (3.1) have not been unambiguously successful, and in order to provide a framework which permits a more convincing derivation, several models of both strong and weak interactions have recently been proposed [11-14] in which J_μ contains pion terms. Up to the term linear in the pion field (which may be either the only pionic contribution, or else the leading member of an infinite series), these models replace the A_μ given

(*) In order for an i-vector to occur, the non-strange strongly interacting parts of J_μ would have to transform as spinors in i-space.

(9) M. L. Goldberger and S. B. Treiman: *Phys. Rev.*, **110**, 1178 (1958); **111**, 354 (1958).

(10) R. F. Sawyer: *Phys. Rev.*, **116**, 231 (1959).

(11) J. Bernstein, S. Fubini, M. Gell-Mann and W. Thirring: preprint (1960)

(12) M. Gell-Mann and M. Lévy: *Nuovo Cimento*, **16**, 705 (1960).

(13) F. Gürsey: *Nuovo Cimento*, **14**, 230 (1960).

(14) F. Gürsey: preprint (1960); to appeat in Annals of Physics.

in (3.1) by

$$A_\mu = -\left\{\bar{\psi}\gamma_\mu\gamma_5\tau_+\psi + \frac{\sqrt{2}\,m}{g}\,\partial_\mu\varphi_+\right\}, \tag{3.2}$$

where m is the nucleon mass and g the PS-PS coupling constant. The observed charged pion decay rate is actually that which would be obtained from the second term *alone*, with neglect of all further corrections.

The effects on $\mathscr{V}$ of the pionic term in (3.2) are therefore interesting already from a fundamental viewpoint. But if $\mathscr{V}$ is subsequently to be calculated on the Chew-Low model, then there exists also a completely different and more elementary reason for including such a contribution to A_μ: On this model, virtual pair creation by the pions is neglected; thus there is no mechanism left whereby A_μ in its conventional form (3.1) could induce pion decay. The simplest phenomenological way of removing this inconsistency is to rewrite A_μ in the form (3.2), appealing to the observed pion decay rate to determine the coefficient of the second term. (As far as $\mathscr{V}$ is concerned, this can be regarded as a simple way of compensating for the omission of those diagrams for instance in which the four-nucleon terms in $\mathscr{L}$ attach a closed loop to one of the external nucleon lines, one at least of the exchanged pions then originating from this loop.)

It so happens that even in a fully relativistic PS-PV theory with A_μ given by (3.2) there is a cancellation (to within a small correction of order μ/m) of all diagrams contributing to pion decay, other than the diagram featuring just the second term of A_μ by itself ([10-12]). Therefore in our particular case the Chew-Low model's neglect of virtual pairs introduces no error as regards pion decay, and the two reasons we have given for including the pion current in A_μ become precisely complementary in the following sense. If this term is already present in the basic current A_μ, we need merely retain it for A_μ to continue to yield the correct pion decay rate in the Chew-Low model; if it is not present in A_μ, it still remains exactly the expression to be used phenomenologically in order to reproduce the decay rate within the model.

Whichever viewpoint is adopted, the relative phase of the two terms in A_μ proves to be important (*). The models mentioned above lead to (3.2) directly, but if the pion term is regarded as compensating for pair suppression in the Chew-Low model, then even though its magnitude may be taken from observation, its phase relative to the nucleon current must be calculated. Again it turns out to be as written in (3.2), according both to simple perturbation theory (which of course completely fails to yield the right magnitude), and to more sophisticated methods ([10]).

(*) In practice, it turns out that as long as they are relative real, a phase difference of π is without consequence.

4. – Effect of the linear pion current on $\mathscr{V}$.

4·1. *Introduction.* – If the vector part V_μ of the self-interacting current is conserved, then the new PNC terms introduced into the Lagrangian by adding $\partial_\mu \boldsymbol{\varphi}$ to A_μ vanish, since by one partial integration we can transform them as follows:

$$V_\mu \partial_\mu \boldsymbol{\varphi} \to -(\partial_\mu V_\mu)\boldsymbol{\varphi} = 0 \,. \tag{4.1}$$

The rest of this section is therefore concerned with the effect of the linear pion current in the case that the mesonic part of V_μ is absent. It will be convenient to start by giving a consecutive discussion of what we shall call the conventional case, in which J_μ is the + component of a vector in i-space:

$$J_\mu^+ = \left\{ \overline{\psi} i \gamma_\mu (1 + i\gamma_5) \tau_+ \psi + \frac{\sqrt{2}\, m}{g} \partial_\mu \varphi_+ \right\} . \tag{4.2}$$

This is the analogue of the case treated by Blin-Stoyle. Once it has been dealt with it will be trivial to indicate the changes that must be made in $\mathscr{V}$ if neutral currents are also included in $\mathscr{L}$.

We saw in the last section that there are no one-pion-exchange terms in $\mathscr{V}$. This makes it convenient to transform the PNC Lagrangian density to a form which is already bilinear in the pion fields. Dropping all PC terms, as well as the PNC four-nucleon contact terms, we have

$$\mathscr{L} = \frac{G\sqrt{2}\, m}{g} \{ \overline{\psi} i \gamma_\mu \tau_+ \psi \partial_\mu \varphi_- + \overline{\psi} i \gamma_\mu \tau_- \psi \partial_\mu \varphi_+ \} \,. \tag{4.3}$$

After one partial integration, the field equations

$$\begin{cases} (\gamma_\mu \partial_\mu + m)\psi + g(\boldsymbol{\tau}\cdot\boldsymbol{\varphi})\gamma_5 \psi = 0 \,, \\ \overline{\psi}(-\gamma_\mu \partial_\mu + m) + g\overline{\psi}(\boldsymbol{\tau}\cdot\boldsymbol{\varphi})\gamma_5 = 0 \,, \end{cases} \tag{4.4}$$

enable us to rewrite $\mathscr{L}$ in the form

$$\mathscr{L} = -\frac{iG\sqrt{2}\, m}{g} \cdot g \cdot \{ \overline{\psi}\gamma_5 [\boldsymbol{\tau}\cdot\boldsymbol{\varphi}, \tau_+]\psi\varphi_- + \overline{\psi}\gamma_5 [\boldsymbol{\tau}\cdot\boldsymbol{\varphi}, \tau_-]\psi\varphi_+ \} \,.$$

By virtue of

$$[\boldsymbol{\tau}\cdot\boldsymbol{\varphi}, \boldsymbol{\tau}] = 2i\,\boldsymbol{\tau}\times\boldsymbol{\varphi} \tag{4.5}$$

this becomes

$$\mathscr{L} = -2G\sqrt{2}\, mi\bar{\psi}\gamma_5(\tau_+\varphi_- - \tau_-\varphi_+)\varphi_0\psi\,. \tag{4.6}$$

The cancellation of the strong coupling constant g could be thought of as just a numerical coincidence, though in the light of the remarks made in the last section it probably has deeper significance.

Because $\bar{\psi}\gamma_5\psi$ lacks a straightforward static limit it is convenient to transform $\mathscr{L}$ once more, bringing it into a form closely similar to that which obtains in the conserved vector current case. To this end we add to $\mathscr{L}$ the four-divergence

$$\Lambda = \sqrt{2}\, Gi\partial_\mu\{\bar{\psi}\gamma_\mu\gamma_5(\tau_+\varphi_- - \tau_-\varphi_+)\varphi_0\psi\}\,; \tag{4.7}$$

by using the field equations we find

$$\mathscr{L} = \sqrt{2}\, Gi\,\{\bar{\psi}\gamma_\mu\gamma_5\tau_+\psi\partial_\mu(\varphi_-\varphi_0) - \bar{\psi}\gamma_\mu\gamma_5\tau_-\psi\partial_\mu(\varphi_+\varphi_0)\}\,. \tag{4.8}$$

In going over to the Hamiltonian formalism we retain only terms to first order in G; then in the static limit the PNC Hamiltonian is given by

$$\mathscr{H} = \sqrt{2}\, Gi\underline{\sigma}\cdot\{\tau_+(\varphi_-\underline{\nabla}\varphi_0 + \varphi_0\underline{\nabla}\varphi_-) - \tau_-(\varphi_+\underline{\nabla}\varphi_0 + \varphi_0\underline{\nabla}\varphi_+)\}\,. \tag{4.9}$$

This can be compared to the corresponding expression in the case of a conserved vector current, which is

$$\mathscr{H}' = \sqrt{2}\, Gi\underline{\sigma}\cdot\{\tau_+(\varphi_-\underline{\nabla}\varphi_0 - \varphi_0\underline{\nabla}\varphi_-) - \tau_-(\varphi_+\underline{\nabla}\varphi_0 - \varphi_0\underline{\nabla}\varphi_+)\}\,. \tag{4.10}$$

Before proceeding to evaluate $\mathscr{V}$, we recall that this merely amounts to integrating the pion field variables out of the two nucleon problem to successively higher order in the (PC) pion-nucleon coupling constant f. A particularly important consequence of this observation is that the transformation properties in i-space of $\mathscr{H}$ and $\mathscr{V}$ must be the same.

The pion-nucleon coupling we use is just the conventional one for the static model:

$$H = \frac{\sqrt{4\pi}\, f}{\mu}\,\underline{\sigma}\boldsymbol{\tau}\cdot\underline{\nabla}\boldsymbol{\varphi}\,. \tag{4.11}$$

No cut-off has been imposed anywhere above, in view of the fact, which has been discussed already, that virtual mesons of high momentum affect the behaviour of $\mathscr{V}$ only at small internucleon distances, at which the wavefunction vanishes because of the hard core. Since $\mathscr{V}$ is eventually to be used in a perturbation calculation its behaviour inside the core is irrelevant.

4·2. *Evaluation of* $\mathscr{V}$. – The explicit evaluation to order f^2 of the $\mathscr{V}$ resulting from (4.9) is straightforward, since we already know that both the pions emitted at the PNC vertex by one nucleon must be absorbed by the other. It turns out that to this order $\mathscr{V}$ vanishes, a result which at first sight is perhaps a little surprising in view of the superficially close similarity of $\mathscr{H}$ to $\mathscr{H}'$, it being known that in the conserved vector current case the latter leads to a non-vanishing $\mathscr{V}'$ by an exactly parallel procedure.

The reason for the cancellation lies in the i-space transformation properties of $\mathscr{H}$: this, after a symbolic partial integration, can be expressed succinctly as

$$\mathscr{H} = G\underline{\sigma}\varphi_0[(\underline{\nabla}\boldsymbol{\tau})\times\boldsymbol{\varphi}]_0 \,. \tag{4.12}$$

Thus $\mathscr{H}$ is the 00 component of a second rank tensor whose trace vanishes by virtue of the identity

$$\boldsymbol{\varphi}\cdot\{\underline{\nabla}\boldsymbol{\tau})\times\boldsymbol{\varphi}\} \equiv 0 \,. \tag{4.13}$$

Therefore $\mathscr{V}$ must also be the 00 component of such an (irreducible) tensor, in which the isotopic dependence lies entirely in the i-spin matrices, $\boldsymbol{\tau}^{(1)}$ and $\boldsymbol{\tau}^{(2)}$, of the two nucleons. The only such tensor is

$$T_{ij} = \{\delta_{ij}\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)} - 3\tau_i^{(1)}\tau_j^{(2)}\} \,, \tag{4.14}$$

and T_{00} cannot occur in the f^2 contributions to $\mathscr{V}$. To see this, let us label by (1) the nucleon which emits the two pions by means of $\mathscr{H}^{(1)}$. To order f^2 there can be no pionic corrections to the PNC vertex, so that only $\tau_+^{(1)}$ and $\tau_-^{(1)}$ can occur in $\mathscr{V}$, but not $\tau_0^{(1)}$. But T_{00} does contain $\tau_0^{(1)}$, whence to this order $\mathscr{V}$ must vanish altogether, as indicated by the explicit calculation (*).

The leading terms in $\mathscr{V}$ are therefore of two distinct types. There are two-pion-exchange terms arising from radiative corrections, and three-pion-exchange terms. Both types are of order f^4, while the leading terms in the conserved vector current case are of order f^2.

At this point we observe that in the analogous problem of ordinary (PC) internucleon forces, the one-pion and two-pion-exchange contributions are actually of comparable importance, because the two-pion terms although prefaced by a smaller constant ($f^2 \approx 0.1$) dip much more steeply just outside the hard core and dominate in its immediate vicinity. Since the situation

(*) For completeness, note that no such cancellation need (or indeed does) occur in $\mathscr{V}'$, since $\mathscr{H}'$ can be written as $G\underline{\sigma}\cdot\{\boldsymbol{\tau}\cdot(\boldsymbol{\varphi}\times\underline{\nabla}\boldsymbol{\varphi}) - \tau_0(\boldsymbol{\varphi}\times\underline{\nabla}\boldsymbol{\varphi})_0\}$. This is a mixture reducible to a scalar and a second rank symmetric traceless tensor, whose relative proportions in any given form in $\mathscr{V}'$ can be determined only by explicit calculation.

in our case is similar, we reach two crucial conclusions. First, of the two types of terms of order f^4, those arising from two-pion exchange are likely to be negligible in their effects compared to the three-pion terms. (This presumes, according to the spirit if not the letter of the static model, that integrals over virtual pion momenta are of order unity, so that an integral over an extra momentum does not compensate the accompanying factor of f^2. In the Appendix we shall make an equivalent assumption by supposing that successively higher order renormalization terms decrease in importance.) Hence we neglect such two-pion terms in the following, and concentrate solely on the three-pion-exchange contributions. Second, again because of the steeper drop near the origin, the fact that the leading terms in $\mathscr{V}$ are of order f^4 does not necessarily imply smaller PNC effects than are predicted if the vector current is conserved and the leading terms are of order f^2. When the three-pion terms have been calculated we shall present some rough numerical estimates in support of this conjecture.

The calculation of the three-pion-exchange terms is a tedious matter which we present only in outline. One could use either a purely diagrammatic method or else apply a sequence of unitary transformations which eliminate the (PC) couplings of the two nucleons to successively higher orders in the meson field. In practice we adopt a compromise approach in which the PC coupling of one of the nucleons, say nucleon (2), $H^{(2)}$, is transformed away in favour of a c-number term and a two-meson interaction, denoted by $H_0^{(2)}$ and $H_2^{(2)}$ respectively. Thus nucleon (2) can emit or absorb pions only in pairs. The transform of the PNC coupling of nucleon (1), $\mathscr{H}^{(1)}$, is given by

$$\mathscr{H}^{(1)} = \mathscr{H}^{(1)} + \delta\mathscr{H}^{(1)} ,$$

where $\delta\mathscr{H}^{(1)}$ is a PNC vertex which can emit or absorb pions singly, but has a one-pion-exchange already built into it in c-number form.

It can then be shown that the only three-pion-exchange PNC diagrams to order f^4 are those of the following general type: nucleon (1), say, emits a pion via $\delta\mathscr{H}^{(1)}$; this pion is reflected by $H_2^{(2)}$, the reflected pion being finally absorbed by the q-number part of the transform of $H^{(1)}$, which is just the original $H^{(1)}$ itself. A further simplification results from the known form (4.14) of the i-spin factor of $\mathscr{V}$, which makes it sufficient to evaluate only the part proportional to $\tau_y^{(1)}\tau_y^{(2)}$ and then symmetrize the result.

One finds finally

$$\mathscr{V} = -\frac{(4\pi)^2 f^4 G}{\mu^4}\frac{1}{8} T_{00}(\underline{\sigma}^{(1)}\times\underline{\sigma}^{(2)}\cdot\underline{\hat{R}})\frac{4F'}{R}(F''G' + F'G'') . \tag{4.15}$$

Here, T_{00} is given by (4.14), $\underline{R} = \underline{r}_2 - \underline{r}_1$, $\underline{\hat{R}}$ is the unit vector along $\underline{R}$; the primes denote differentiation with respect to R, and F and G are functions

of R alone, defined by

$$F = \sum_{\underline{k}} \exp[i\underline{k}\cdot\underline{R}]/\omega_k^2 = (4\pi R)^{-1}\exp[-\mu R]\,, \tag{4.16}$$

$$G = \sum_{\underline{k}} \frac{\exp[i\underline{k}\cdot\underline{R}]}{\omega_k^3} = (2\pi^2 R)^{-1}\int_0^\infty k(k^2+\mu^2)^{-\frac{3}{2}}\sin(kR)\,\mathrm{d}k\,. \tag{4.17}$$

By using the representation

$$\int_0^\infty (x^2+a^2)^{-\frac{1}{2}}\cos x\,\mathrm{d}x = K_0(a) = (i\pi/2)H_0^{(1)}(ia)\,, \tag{4.18}$$

and the recurrence relations for Bessel functions, it is possible after some manipulation to rewrite $\mathscr{V}$ as

$$\mathscr{V} = -f^4 G\mu^3(2\pi)^{-2}T_{00}(\underline{\sigma}^{(1)}\times\underline{\sigma}^{(2)}\cdot\underline{\hat{R}})\,\lambda(x)\,, \tag{4.19}$$

where $x = \mu R$ and

$$\lambda(x) = \exp[-2x]x^{-8}(1+x)\{(\pi i/2)H_0^{(1)}(ix)[2+2x+x^2+x^3]x^4 + \\ + (\pi/2)H_1^{(1)}(ix)[2+2x+3x^2+7x^3+7x^4+2x^5-2x^6]\}\,. \tag{4.20}$$

For comparison we quote the leading term in the conserved vector current case as given by Blin-Stoyle:

$$\begin{cases} \mathscr{V}' = -f^2 G\mu^3(2\pi)^{-1}\{\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)} - \tau_0^{(1)}\tau_0^{(2)}\}[\underline{\sigma}^{(1)}\times\underline{\sigma}^{(2)}\cdot\underline{\hat{R}}]\lambda'(x)\,, \\ \lambda'(x) = \exp[-2x]x^{-6}(1+x)^2\,. \end{cases} \tag{4.21}$$

The asymptotic form of $\lambda(x)$ as $x\to\infty$ is

$$\lambda(x)\sim(\pi/2x)^{\frac{1}{2}}\exp[-3x]\{1-x^{-1}/8+O(x^{-2})\}\,, \tag{4.22}$$

while as $x\to 0$ one has

$$\lambda(x)\sim -2x^{-9}\,. \tag{4.23}$$

Both these asymptotic forms are actually somewhat misleading. Detailed inspection shows that on the one hand the (only) zero of $\lambda(x)$ occurs far enough out for the positive tail to be quantitatively insignificant; on the other hand the hard core radius is sufficiently large to prevent the wavefunction from penetrating into the region where (4.23) is a good approximation. Furthermore, even if the behaviour of $\lambda(x)$ at such small distances were of interest, one would have to employ a cut-off to make it at all meaningful.

4·3. *Comparison of $\mathscr{V}$ and $\mathscr{V}'$.* – The object of this subsection is to substantiate the conjecture made above that the steeper drop at small distances of $\mathscr{V}$ as compared with $\mathscr{V}''$ can, at least in part, compensate the smaller strength constant. A similar mechanism may conceivably boost the effective strength of the PNC potential relative to the PC nuclear forces, in which case previous estimates [2] of the expected magnitude of PNC effects in nuclear systems might have to be revised upward. At this time however we confine ourselves entirely to a comparison of $\mathscr{V}$ and $\mathscr{V}'$, since even this simpler task is too complex to allow more than a rough plausibility check on our assertion.

We intend to form a representative estimate of the relative amplitudes of parity impurities admixed into nuclear states by $\mathscr{V}$ and $\mathscr{V}''$. Both these potentials are so weak that a first order perturbation treatment would certainly be adequate if only we knew the nuclear wavefunctions spanning the unperturbed states of sharp parity. In the absence of such information we envisage, for definiteness, a fairly heavy nucleus and argue as follows. The amplitudes of admixture are likely to be governed by the two-body matrix elements of $\mathscr{V}$ ($\mathscr{V}''$) taken between opposite parity states of a pair of nucleons colliding in nuclear matter. The relative wave function of the nucleons vanishes inside the hard core, and approaches the unperturbed form with a healing distance of roughly k_F^{-1}, k_F being the Fermi momentum. We ignore all complications due to spin and i-spin operators, and confine ourselves to evaluating the following radial integrals:

$$\begin{Bmatrix} I\,\{f^2/2\pi\}^{-2} \\ I'\{f^2/2\pi\}^{-1} \end{Bmatrix} = \int_{r_c}^{\infty} r^2\,\mathrm{d}r\, j_0(Kr) j_1(Kr)\,(1-\exp[-k_F(r-r_c)])^2 \begin{Bmatrix} \lambda\,(\mu r) \\ \lambda'(\mu r) \end{Bmatrix}. \tag{4.24}$$

The integrands are constructed as follows: The two nucleons with relative energy $K^2/2m$ come together in an S-wave ($j_0(Kr)$) and separate in a P-wave ($j_1(Kr)$), both the initial and final functions being modified by a factor $(1-\exp[-k_F(r-r_c)])$ outside the hard core (radius r_c) and vanishing inside it. We are neglecting the modification of the relative wavefunction by the attractive PC forces; these will increase the wavefunction at small distances so that we are actually underestimating $\mathscr{V}$ relative to $\mathscr{V}'$.

For our purpose, only the ratio ϱ of I to I' is relevant. ϱ is actually a function of (K/μ), (k_F/μ), and (μr_c); since it is far more sensitive to (μr_c) than to the other variables we have, for this first orientation, fixed $K = k_F = 2\mu$. We also adopt the value 0.08 for f^2. We have evaluated I and I' numerically for these chosen values of K and k_F and for three different values of $x_0 = \mu r_c$. The results are given in Table I.

The current value of r_c is in the vicinity of $0.4 \cdot 10^{-13}$ cm. By inspecting Table I, and recalling that it actually underestimates ϱ, we conclude, in the

TABLE I.

$x_0 = \mu r_c$	r_c (10^{-13} cm)	$I\{f^4/(2\pi)^2\}^{-1}$	$I'\{f^2/2\pi\}^{-1}$	$\varrho = I/I'$
0.2	0.29	7.05	0.141	0.635
0.3	0.43	1.78	0.090 2	0.251
0.4	0.57	0.728	0.063 6	0.145

case of charged currents only, that even if the vector current is not conserved, it appears likely that amplitude-dependent PNC effects in nuclei remain between 0.3 and, say, 0.5 times as strong as they would be in the conserved vector current case.

5. – Other forms of self-interacting currents.

So far we have been concerned exclusively with the case where the self-interacting current J_μ is the +component of an i-vector, so that $\mathscr{L}$ is of the form

$$\mathscr{L} \sim J_\mu^x J_\mu^{x*} + J_\mu^y J_\mu^{y*} . \tag{5.1}$$

This corresponds to charged intermediate boson fields without neutral counterparts.

LEE and YANG [7] have recently proposed a detailed hypothesis concerning the intermediate bosons, in which neutral components play an essential role. For our purpose it suffices to note that when this scheme is translated into a language from which the intermediate field has been eliminated, it corresponds, as far as the non-strange strongly interacting fields are concerned, to substituting for $\mathscr{L}$ as given by (5.1) the i-scalar

$$\mathscr{L} \sim \boldsymbol{J}_\mu \cdot \boldsymbol{J}_\mu^* . \tag{5.2}$$

In the conserved vector current case one now has

$$\mathscr{H}' = G\underline{\sigma}\{\boldsymbol{\tau}\cdot\boldsymbol{\varphi} \times \underline{\nabla}\boldsymbol{\varphi}\} , \tag{5.3}$$

which in lowest order leads to

$$\mathscr{V}' = -\frac{f^2 G \mu^3}{2\pi} (\boldsymbol{\tau}^{(1)}\cdot\boldsymbol{\tau}^{(2)})(\underline{\sigma}^{(1)} \times \underline{\sigma}^{(2)} \cdot \hat{R})\, \lambda'(x) . \tag{5.4}$$

In the opposite case one must make a similar replacement in (4.12) and finds

the remarkable cancellation

$$\mathscr{H} = G\underline{\sigma}\,\{\boldsymbol{\varphi}\cdot(\underline{\nabla}\boldsymbol{\tau})\times\boldsymbol{\varphi}\} \equiv 0\,. \tag{5.5}$$

A corresponding identity holds already for the Lagrangian density, so that with an i-scalar $\mathscr{L}$ of this type addition of a linear pion term to J_μ has no effect whatsoever on $\mathscr{V}$ irrespective of whether the vector current is conserved.

We conclude that in this scalar case the SIC hypothesis can be tested only if the conserved vector current hypothesis also holds.

* * *

Most of this work was done while the author was summer visitor at the University of Washington in Seattle. He welcomes this opportunity to acknowledge several helpful comments by Professor E. M. Henley, and to thank the Physics Department for its hospitality. He is grateful also to Professor J. R. Oppenheimer for extending to him the support and hospitality of the Institute for Advanced Study.

Appendix

If the fundamental (PC) nucleon-pion coupling is believed to be the PS one then the use of the static interaction (4.11) in conjunction with other couplings of pions and nucleons, either to each other or to different fields, raises some problems of principle which have been treated in a different context elsewhere ([8]). In brief, (4.11) must in that case be regarded as resulting from a unitary transformation (the Foldy-Dyson equivalence transformation) of the entire Hamiltonian with PS-PS coupling. To a first approximation (4.11) then replaces the usual PS-PS interaction; but if other interaction terms are present then these must also be transformed simultaneously, and any new parts of their transforms included in the total Hamiltonian in which the strong nucleon-pion coupling is represented by (4.11).

In the case of the pionic renormalization of non-pionic vertices, these new terms transformed into the Hamiltonian are found to lead to physically unacceptable results, making the use of the static model suspect, at least in such contexts.

The present situation is somewhat different, since we are specifically *not* interested in questions of renormalization, and desire only to obtain the leading terms of a two-nucleon potential. This difference in orientation naturally does not alleviate the difficulties mentioned above, but at least it enables one to shelve them while dealing with the particular problem in hand. In the spirit of these remarks, the only question relevant to $\mathscr{V}$ to order f^4 is whether the transform of $\mathscr{H}$ or $\mathscr{H}'$ contains new terms which couple the nucleons to expres-

sions up to cubic in the pion fields. Other types of coupling can contribute only to the renormalization of terms already present, or to four-pion-exchange.

On performing the transformation in the standard way (*) in either its strong or its weak form, it appears at once that no such terms enter, the first new types of coupling which survive in the limit of static nucleons being already quartic in q. Hence as long as we are not trying to calculate renormalization effects the equivalence transformation introduces no new features into $\mathscr{V}$ or $\mathscr{V}'$.

(*) Since the equivalence transformation involves γ_5 in a crucial way, it must of course be applied to the full Hamiltonian resulting from $\mathscr{L}$, and not only to its static limit. The static limit must be taken again after the full transform has been obtained.

Notes added in proof.

1) New evidence in favour of the conserved vector current, postdating that of ref. (6), has been given by M. E. Nordberg, F. B. Morinigo and C. A. Barnes: *Phys. Rev. Lett.*, **5**, 321 (1960).

2) The author has been informed privately that the vanishing of $\mathscr{V}$ to order f^2 had already been noted by R. J. N. Phillips (unpublished).

RIASSUNTO (*)

Se la debole densità lagrangiana $\mathscr{L}$ consiste di correnti cariche autointeragenti, l'inclusione fra queste di un termine lineare nel campo pionico porta ad un potenziale internucleonico statico $\mathscr{V}$ che non conserva la parità, ed ha effetti il cui ordine di grandezza generico non dipende più dai termini pionici quadratici necessari per la conservazione della corrente vettoriale. Se anche correnti neutre contribuiscono ad $\mathscr{L}$ in modo da renderlo uno scalare isotopico, l'inclusione di termini lineari pionici non ha effetti su $\mathscr{V}$, che rimane inapprezzabile a meno che la corrente vettoriale sia conservata. Si discutono le origini di tali termini pionici; ne risulta incidentalmente che in conseguenza dell'invarianza rispetto all'inversione del tempo, $\mathscr{V}$ non può contenere contributi derivanti dallo scambio di un singolo pione.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XIX, N. 3 1° Febbraio 1961

Closed Formulae for the Statistical Weights.

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(ricevuto il 10 Ottobre 1960)

Summary. — The branching ratios between the different charge distributions of mesons emerging with a given multiplicity from a high-energy collision are computed from isospin conservation and a statistical hypothesis on the probability of the possible end states. The method uses the Wigner-type of projection operator in group space and yields a closed formula, which is entirely general, for the weights of the different charge states. The formula is especially simple in the case of n isospin 1 particles coupled to a low total isospin. Cases with n isospin 1 particles and one or two isospin $\frac{1}{2}$ particles are simply related to the former result.

In the analysis of multiple production experiments from high energy collisions, the need arises to know the branching ratio between channels with the same multiplicity but different charges of the end particles. *E.g.* between the reactions

$$\mathrm{p} + \mathrm{p} \to \begin{cases} \mathrm{pp}\pi^+\pi^- \\ \mathrm{pp}\pi^0\pi^0 \\ \mathrm{pn}\pi^+\pi^0 \\ \mathrm{nn}\pi^+\pi^+ \end{cases}$$

If the following assumptions are made

a) conservation of isospin and charge;

b) equal weight or each possible isospin end state with fixed T and T_3 (« statistical hypothesis »)

it is possible to compute those branching ratios.

Indeed, one knows the total isospin $T^{(i)}$ and $T_3^{(i)}$ from the initial state of the colliding particles. If we choose an orthonormal system of base vectors $|T, T_3, \alpha\rangle$ in which to describe our isospin end states, then the state after the collision is given—in isospace—by

$$\sum_{\alpha} c_\alpha |T^{(i)}, T_3^{(i)}, \alpha\rangle \,. \tag{1}$$

Here α stands for all additional quantum numbers required to describe the n-particle state completely in isospin space. But the system of the $|T, T_3, \alpha$ is not the one used by the experimentalist who measures the end products of a collision. He describes the nature of the final particles and their charges, his measurement states that the state vector of the system in isospace is of the form

$$|t_1\theta_1, t_2\theta_2, \ldots, t_n\theta_n\rangle \,, \tag{2}$$

where: t_i: isospin of the i-th particle,

$\theta_i = (t_3)_i =$ 3-rd component of $\boldsymbol{t}_i$.

Assumptions *a*) and *b*) mean that the transition matrix element squared

$$|\langle T^{(f)}, T_3^{(f)}, \alpha^{(f)}; \pi^{(f)} | \mathscr{T} | T^{(i)}, T_3^{(i)}, \alpha^{(i)}; \pi^{(i)}\rangle|^2$$

is independent of $\alpha^{(f)}$ ($\pi^{(i)}$ and $\pi^{(f)}$ represent the quantum numbers for the state vector in ordinary space), and vanishes for $T^{(f)} \neq T^{(i)}$ and $T_3^{(f)} \neq T_3^{(i)}$. Because the initial state is a two-particle state $T^{(i)}$ and $T_3^{(i)}$ suffice to define it in isospace. The probability for state (2) is therefore in general

$$\begin{aligned} &|\langle t_1\theta_1, \ldots, t_n\theta_n; \pi^{(f)} | \mathscr{T} | T, T_3 \pi^{(i)}\rangle|^2 \\ &= |\sum_{\alpha} \langle t_1\theta_1, \ldots, t_n\theta_n | T, T_3, \alpha\rangle \langle T, T_3\alpha; \pi^{(f)} | \mathscr{T} | T, T_3; \pi^{(i)}\rangle|^2 \end{aligned} \tag{3}$$

but because of *b*)

$$= F(\pi^{(f)}) \sum_{\alpha} |\langle t_1\theta_1, \ldots, t_n\theta_n | T, T_3, \alpha\rangle|^2 \,, \tag{4}$$

where $F(\pi^{(f)})$ is a function depending on the nature of the particles produced, on the configuration in p-space, and also of course on the initial condition, before the collision, but not on $\alpha^{(f)}$.

We see that the probability of certain charges $\theta_1, ..., \theta_n$ will be given by

$$(5) \qquad P^{(T)}_{\theta_1, ..., \theta_n} \left[\sum_{\alpha} |\langle t_1\theta_1, ..., t_n\theta_n | T, T_3, \alpha \rangle|^2\right] \left[\sum_{\theta_1, ..., \theta_n} \sum_{\alpha} |\langle t_1\theta_1 ... | T, T_3 \alpha \rangle|^2\right]^{-1}$$

and this is independent of the configuration in p-space.

The method used up to now (*) to compute $P^{(T)}_{\theta_1, ..., \theta_n}$, was to choose explicitly a complete orthonormal system $|T, T_3, \alpha\rangle$, and to carry the summation $\sum_{\alpha}$ out, using of course the special properties of the system chosen to introduce simplifications whenever possible.

What one does in fact is to compute the projection of $|t_1\theta_1, ..., t_n\theta_n\rangle$ on all base vectors of $|T, T_3, \alpha\rangle$ and build the sum of the squares. This is nothing but the square of the projection of $|t_1\theta_1, ..., t_n\theta_n\rangle$ on the subspace spanned by the base $|T, T_3, \alpha\rangle$ (for T, T_3 given). Consequently, if one could compute directly the length of this projection, one could dispense with the computation of the components.

The vectors $|t_1\theta_1, ..., t_n\theta_n\rangle$ span in fact a representation space of the rotation group. If one performs a rotation (defined by the Euler angles α, β, γ) these vectors transform according to

$$(6) \qquad |t_1\theta_1, ..., t_n\theta_n\rangle = \sum_{(\theta')} \mathscr{D}^{t_1}_{\theta_1\theta'_1}(\alpha\beta\gamma)\, \mathscr{D}^{t_2}_{\theta_2\theta'_2}(\alpha\beta\gamma) ... \mathscr{D}^{t_n}_{\theta_n\theta'_n}(\alpha\beta\gamma)\, |t_1\theta'_1, ..., t_n\theta'_n\rangle$$

and the matrices

$$(7) \qquad D_{\theta_1, \theta_2 ...; \theta'_1, \theta'_2 ...}(\alpha\beta\gamma) \equiv \mathscr{D}^{t_1}_{\theta_1, \theta'_1}(\alpha\beta\gamma) ... \mathscr{D}^{t_n}_{\theta_n, \theta'_n}(\alpha\beta\gamma)$$

form a representation of the rotation $(\alpha\beta\gamma)$.

This representation can be reduced into a direct sum of irreducible representations, by a suitable transformation. The reduction is equivalent to finding the invariant subspaces in the space of $|t_1\theta_1, ..., t_n\theta_n\rangle$. In particular to the irreducible representation of degree $2T+1$ corresponds an invariant subspace of dimension $2T+1$ whose base vectors are transformed among themselves by any rotation $(\alpha\beta\gamma)$. The vectors $|T, T_3, \alpha\rangle$ belong necessarily to such a subspace. There may be more than one subspace of dimension $2T+1$, because the representation in the space of the $|t_1\theta_1, ..., t_n\theta_n\rangle$ may contain more than

(*) F. CERULUS: *Suppl. Nuovo Cimento*, **15**, 402 (1960); here a coupling scheme is used where the $(i+1)$-st particle is coupled to the i previous ones. This leads to recurrence formulae involving Clebsch-Gordan coefficients. A. PAIS: *Ann. of Phys.*, **9**, 548 (1960); the α are quantum numbers characterizing the symmetry of the wave functions. For $T=0$ and $T=1$, this is sufficient to define a complete base for many-pion states.

once the irreducible representation equivalent to $\mathscr{D}^T$. Let us call the direct sum of these subspaces $\Gamma^{(T)}$.

We now recall the form of the projection operator of a vector from the $|t_1\theta_1, ..., t_n\theta_n\rangle$ space onto the subspace $\Gamma^{(T)}$.

$$P^{\Gamma^{(T)}} = \frac{2T+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} d\beta \sin\beta\, \chi^{(T)*}(\alpha\beta\gamma)\, D_{\theta_1\theta_2...;\theta'_1\theta'_2...}\,, \tag{8}$$

where $(1/8\pi^2)\int d\alpha \int d\gamma \int d\beta \sin\beta$ is the invariant integral in group space $\chi^{(T)}(\alpha\beta\gamma)$ is the character of the rotation $(\alpha\beta\gamma)$ in the representation of dimension $2T+1$.

$D_{\theta_1\theta_2...;\theta'_1\theta'_2...}$ is the rotation operator in the space of $|t_1\theta_1, ..., t_n\theta_n\rangle$, and is given by (7).

(8) is the application to the rotation group of a general theorem on such projection operators for finite groups or continuous groups which allow an invariant integration in group space (*).

To get the square of the projection of $|t_1\theta_1, ..., t_n\theta_n\rangle$ on $\Gamma^{(T)}$ we build simply

$$|P^{\Gamma^{(T)}}|t_1\theta_1, ..., t_n\theta_n\rangle|^2 = \langle t_1\theta_1, ..., t_n\theta_n\ P^{\Gamma^{(T)}}\ t_1\theta_1, ..., t_n\theta_n$$

because for any projection operator $P^2 = P$ and P is hermitian.
Consequently

$$P^{(T)}_{\theta_1\theta_2...\theta_n} = \frac{2T+1}{8\pi^2}\int_0^{2\pi} d\alpha \int_0^{2\pi} d\gamma \int_0^{\pi} d\beta \sin\beta\, \chi^{(T)*}(\alpha\beta\gamma)\, \mathscr{D}^{(t_1)}_{\theta'_1\theta_1}(\alpha\beta\lambda) ... \mathscr{D}^{(t_n)}_{\theta'_n\theta_n}(\alpha\beta\gamma)\,. \tag{9}$$

It is now a matter of straightforward computation to find $P^{(T)}_{\theta_1\theta_2...}$ in each case. Indeed, let us consider the expressions for $\chi^{(T)}$ and $\mathscr{D}^{(t)}$:

$$\mathscr{D}^{(j)}_{m,m}(\alpha\beta\gamma) = \left(\frac{1+\cos\beta}{2}\right)^m \boldsymbol{P}^{0,2m}_{T-m}(\cos\beta)\exp[-im(\alpha+\gamma)]\,, \tag{10) (**}$$

$$\chi^{(j)}(\alpha\beta\gamma) = \sum_m \mathscr{D}^{(j)}_{m,m}(\alpha\beta\gamma)\,, \tag{10a}$$

with

$$\boldsymbol{P}^{0,2m}_{T-m}(x) = \text{Jacobi polynomial} = \tag{10b}$$
$$= 2^{-T+m}\sum_{\nu=0}^{T-m}\binom{T-m}{\nu}\binom{T+m}{T-m-\nu}(x-1)^{T-m-\nu}(x+1)^{\nu}\,.$$

(*) E. WIGNER: *Gruppentheorie und Quantenmechanik*, chap. XII.
(**) Cf. *e.g.*: A. R. EDMONDS: *Angular momentum in quantum mechanics*, CERN 55-26.

With these expressions (9) can be transformed into

$$(11)\qquad P^{(T)}_{\theta_1\theta_2,\ldots,\theta_n} = \frac{2T+1}{8\pi^2}\int d\alpha\int d\gamma\int d\beta\,\sin\beta\left(\frac{1+\cos\beta}{2}\right)^{\theta_1+\theta_2+\ldots+\theta_n}\cdot$$
$$\cdot\Big[\prod_{i=1}^{n} \boldsymbol{P}^{0,2\theta_i}_{t_i-\theta_i}(\cos\beta)\exp\left[-i(\theta_1+\theta_2+\ldots+\theta_n)(\alpha+\gamma)\right]\Big]\cdot$$
$$\cdot\left[\sum_{m=-T}^{+T}\left(\frac{1+\cos\beta}{2}\right)^{m}\boldsymbol{P}^{0,2m}_{T-m}(\cos\beta)\exp\left[+im(\alpha+\gamma)\right]\right],$$

$$(12)\qquad = (2T+1)\,2^{-2m-1}\int_{-1}^{+1} dx(1+x)^{2m}\,\boldsymbol{P}^{0,2m}_{T-m}(x)\,\Big[\prod_{i=1}^{n}\boldsymbol{P}^{0,2\theta_i}_{t_i-\theta_i}(x)\Big],$$

where we have used that $(1/2\pi)\int_0^{2\pi}\exp[im\alpha]\,d\alpha = \delta_{0,m}$ (Kronecker δ) and where $m \equiv \theta_1+\theta_2+\ldots+\theta_n = T_3$ and $x = \cos\theta$.

The integrand is a polynomial of degree $N = T + \sum_{i=1}^{n} t_i - 2m$. If we represent it by

$$\sum_{j=0}^{N} a_j x^j$$

then

$$(13)\qquad P^{(T)}_{\theta_1,\ldots,\theta_n} = (2T+1)\,2^{-2m}\sum_{j=0}^{[N/2]}\frac{a_{2j}}{2j+1}.$$

Formula (10) assumes that $m \geq 0$. This is no restriction because one can prove easily (*) that

$$P^{(T)}_{\theta_1,\theta_2,\ldots,\theta_n} = P^{(T)}_{-\theta_1,\ \theta_2,\ldots,\ \theta_n}.$$

It is obvious from the foregoing that $P^{(T)}_{\theta_1,\ldots,\theta_n}$ does not depend on the order of the θ_i.

If one does not label explicitly the particles of the same kind (which are in general distinguishable because they have different momenta), one shall want to sum over the different permutations of the $\theta_1, \ldots, \theta_n$ which give rise to distinguishable states. Permutations which shuffle only particles of the same kind and charge among themselves do not lead to distinguishable states. In order to find then the probability of a certain charge distribution one has to multiply the coefficient $P^{(T)}_{\theta_1,\ldots,\theta_n}$ with the number of permutations of the $\theta_1, \ldots, \theta_n$ which give rise to a reordering of the numbers θ_1 to θ_n. One permutes

(*) F. CERULUS: loc. cit.

of course only the θ's belonging to the same kind of particles. This coefficient we shall denote by ${}^*P^T_{\theta_1,\dots,\theta_n}$.

If *e.g.* we want the weight of the charge distribution of n pions, in isospin T-states, without consideration of their momenta, we get for this

$$\text{Probability for } n_+\,\pi^+,\ n_0\,\pi^0,\ n_-\,\pi^- \equiv {}^*P^{(T)}_{n_+,n_0,n_-} = \frac{n!}{n_+!\,n_0!\,n_-!}\,P^{(T)}_{\underbrace{+\,+\dots}_{n_+}\ \underbrace{0\,0\dots}_{n_0}\ \underbrace{-\,-\dots}_{n_-}}\,.$$

The properties of the $P^{(T)}_{\theta_1,\dots,\theta_n}$ due to the unitarity of the transformation between the vectors $|t_1\theta_1, \dots, t_n\theta_n\rangle$ and $|T, T_3, \alpha\rangle$ have been discussed already (*). Let us recall only that

$$\sum_{m^{(\theta)}_{\text{fixed}}} {}^*P^{(T)}_{\theta_1,\dots,\theta_n} = \varrho^{(T)}\,,$$

where the sum is over all combinations of $\theta_1, \dots, \theta_n$ such that $\theta_1+\theta_2+\dots\theta_n = m$ remains constant. $\varrho^{(T)}$ is the number of independent isospin functions that can be formed to a fixed T and T_3 with the given number of particles having isospin $t_1, t_2, \dots, t_n$ $\varrho^{(T)}$ has already been computed (**).

We shall now consider a few important cases and calculate the formula for each of them.

A) Case of n pions. – Here we have only to consider $\mathscr{D}^1_{m,m}$ in (9)

$$\begin{aligned}
\mathscr{D}^1_{+1,+1}(\alpha\beta\gamma) &= \exp\left[-i(\alpha+\gamma)\right]\tfrac{1}{2}(1+\cos\beta)\\
\mathscr{D}^1_{0,0}(\alpha\beta\gamma) &= \cos\beta \qquad\qquad \chi^{(1)} = \sum_{m=-1}^{+1}\mathscr{D}^1_{m,m}\\
\mathscr{D}^1_{-1,-1}(\alpha\beta\gamma) &= \exp\left[\,i(\alpha+\gamma)\right]\tfrac{1}{2}(1+\cos\beta)
\end{aligned}$$

and (12) reduces to (n_+, n_0, n_- are the numbers of π^+, π^0, π^- respectively; $m = n_+ - n_-$)

$$P^T_{n_+,n_0,n_-} = (2T+1)\,2^{-2n_+-1}\int_{-1}^{+1}(1+x)^{2n_+}x^{n_0}\boldsymbol{P}^{0,2m}_{T-m}(x)\,\mathrm{d}x\,, \tag{14}$$

which is easily expressed as a double sum

$$P^T_{n_+,n_0,n_-} = (2T+1)\,2^{-2n_+-T+m}(-1)^{T-m}\sum_{\nu=0}^{T-m}\binom{T-m}{\nu}\binom{T+m}{T-m-\nu}(-1)^\nu\cdot$$
$$\cdot\left[A(2n_+ + \nu, n_0, T-m-\nu) + (-1)^{n_0}A(T-m-\nu, n_0, 2n_+ + \nu)\right],$$

(*) F. Cerulus: loc. cit.

(**) Y. Yeivin and A. de-Shalit: *Nuovo Cimento*, **1**, 1147 (1955); B. M. Barbašev and V. S. Barašenkov: *Suppl. Nuovo Cimento*, **7**, 19 (1957).

with

$$A(a,b,c) \equiv \sum_{p=0}^{a} \binom{a}{p} \frac{(p+b)!\,c!}{(p+b+c+1)!}\,. \tag{15}$$

Especially:

For $T=0$

$$P^{(0)}_{n_+,n_0,n_-} = \begin{cases} 2^{-2n_+} \sum_{q=0}^{n_+} \binom{2n_+}{2q} \frac{1}{n_0+2q+1}\,, & \text{for } n_0 \text{ even}, \\ 2^{-n_+} \sum_{q=0}^{n_+-1} \binom{2n_+}{2q+1} \frac{1}{n_0+2q+2}\,. & \text{for } n_0 \text{ odd}. \end{cases} \tag{16}$$

For $T=1$

$$\begin{cases} P^{(1)}_{n_+,n_0,n_-} \quad 3P^{(0)}_{n_+,n_0+1,n_-}\,, \\ m = 0\,, \\ P^{(1)}_{n_+,n_0,n_-} = 3P^{(0)}_{n_+,n_0,n_-+1}\,, \\ m = 1\,. \end{cases} \tag{17}$$

B) Case of 2 nucleons and n pions (or 2 K-mesons and n pions). – Because $\mathscr{D}^{\frac{1}{2}}_{m,m} = \exp[im(\alpha+\gamma)]\cos(\beta/2)$ we have, from (9) or (12)

$$P^{(T)}_{m_1,m_2;n_+,n_0,n_-} = (2T+1)\,2^{-(n_++n_-+m)-2} \int_{-1}^{1} (1+x)^{n_++n_-+m+1}\, x^{n_0}\, \boldsymbol{P}^{0,2m}_{T-m}(x)\,\mathrm{d}x\,, \tag{18}$$

as

$$n_+ + n_- + m = 2n_+ + m_1 + m_2 = \begin{cases} 2n_+ + 1\,, & \text{for pp}, \\ 2n_-\,, & \text{for pn}, \\ 2n^+ - 1\,, & \text{for nn}, \end{cases}$$

we find, by comparing (18) with (14), the following relations with the coefficients for n pions only:

$$\begin{cases} P^{(T)}_{pp,n_+,n_0,n_-} = P^{(T)}_{n_++1,n_0,n_-}\,, \\ P^{(T)}_{nn;n_+,n_0,n_-} = P^{(T)}_{n_+,n_0,n_-+1}\,, \\ P^{(T)}_{pn;n_+,n_0,n_-} = \tfrac{1}{2}[P^{(T)}_{n_+,n_0,n_-} + P^{(T)}_{n_+,n_0+1,n_-}]\,. \end{cases} \tag{19}$$

C) Case of one nucleon and n pions. – Formula (12) in this case becomes:

$$P^{(T)}_{m_1;\,n_+,\,n_0,\,n_-} = (2T+1)\,2^{-(n_+ + n_- + m + \frac{3}{2})} \int_{-1}^{+1} (1+x)^{n_+ + n_- + m + \frac{1}{2}}\, x^{n_0}\, \boldsymbol{P}^{0,2m}_{T-m}(x)\,\mathrm{d}x\,.$$

Let us put

$$T = J + \tfrac{1}{2}\,, \qquad m = M + \tfrac{1}{2}\,.$$

Then

$$P^{(T)}_{m_1;\,n_+,\,n_0,\,n_-} = 2(J+1)\,2^{-(2n_+ + m_1 + \frac{3}{2})} \int_{-1}^{+1} (1+x)^{2n_+ + m_1 + \frac{1}{2}}\, x^{n_0}\, \boldsymbol{P}^{0,2M+1}_{J-M}(x)\,\mathrm{d}x =$$

$$= (J+1)\,2^{-(n_+ + n_- + J + 1)} (-1)^{J-M} \sum_{\nu=0}^{J-M} (-1)^{\nu} \binom{J-M}{\nu} \binom{J+M+2}{J-M-\nu},$$

$$[A(2n_+ + 1 + \nu,\, n_0,\, J-M-\nu) + (-1)^{n_0} A(J-M-\nu,\, n_0,\, 2n_+ + 1 + \nu)]\,, \tag{20}$$

with the same definition of A as in (15). Formulae (20) and (15) are very similar, and suggest also in this case simple relations with the coefficients for n pions only.

Especially:

For $\begin{cases} T = \frac{1}{2} \\ m = \frac{1}{2} \end{cases}$ *i.e.* $J = 0$ and $M = 0$,

$$\begin{aligned} P^{(\frac{1}{2})}_{\frac{1}{2};\,n_+,\,n_0,\,n_-} &= P^{(0)}_{n_+,\,n_0,\,n_-} + P^{(0)}_{n_+,\,n_0+1,\,n_-}\,, \\ P^{(\frac{1}{2})}_{-\frac{1}{2};\,n_+,\,n_0,\,n_-} &= 2P^{(0)}_{n_+,\,n_0,\,n_-+1}\,. \end{aligned} \tag{21}$$

For $\begin{cases} T = \frac{3}{2} \\ m = \frac{1}{2} \end{cases}$ *i.e.* $J = 1$ and $M = 1$.

$$\begin{aligned} P^{(\frac{3}{2})}_{\frac{1}{2};\,n_+,\,n_0,\,n_-} &= 3P^{(0)}_{n_+,\,n_0+2,\,n_-} + 2P^{(0)}_{n_+,\,n_0+1,\,n_-} - P^{(0)}_{n_+,\,n_0,\,n_-}\,, \\ P^{(\frac{3}{2})}_{-\frac{1}{2};\,n_+,\,n_0,\,n_-} &= 6P^{(0)}_{n_+,\,n_0+1,\,n_-+1} - 2P^{(0)}_{n_+,\,n_0,\,n_-+1}\,, \end{aligned} \tag{22}$$

For $\begin{cases} T = \frac{3}{2} \\ m = \frac{3}{2} \end{cases}$ *i.e.* $J = 1$, $M = 1$ $\quad n_+ - n_- = 1$ or 2,

$$\begin{aligned} P^{\frac{3}{2}}_{\frac{1}{2};\,n_+,\,n_0,\,n_-} &= 2P^{0}_{n_+,\,n_0,\,n_-+1} + 2P^{0}_{n_+,\,n_0+1,\,n_-+1}\,, \\ P^{\frac{3}{2}}_{-\frac{1}{2};\,n_+,\,n_0,\,n_-} &= 4P^{(\ )}_{n_+,\,n_0,\,n_-+2}\,, \end{aligned} \tag{23}$$

* * *

The author is very much indebted to Professor A. GAMBA for suggesting the use of projection operators in this problem.

Hospitality at CERN is gratefully acknowledged.

RIASSUNTO (*)

Si calcolano i rapporti fra le differenti distribuzioni di carica dei mesoni emergenti con una data molteplicità da una collisione ad alta energia in base alla conservazione dell'isospin e ad una ipotesi statistica sulla probabilità dei possibili stati finali. Il metodo sfrutta l'operatore di proiezione di tipo Wigner nello spazio dei gruppi e fornisce una formula chiusa, del tutto generale, per i pesi dei differenti stati di carica. La formula è specialmente semplice nel caso di n particelle con isospin 1 accoppiate ad un basso isospin totale. Al precedente risultato si riferiscono in modo semplice alcuni casi di n particelle con isospin 1 e una o due particelle con isospin $\frac{1}{2}$.

(*) *Traduzione a cura della Redazione.*

Identities between Strong and Weak Interactions.

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(ricevuto il 10 Ottobre 1960)

Summary. — Models of the axial vector weak interaction proposed by Gell-Mann and Lévy are further investigated. The condition $P^{\alpha}_{\mu,\mu}(x) = ia\pi^{\alpha}(x)$, on the weak axial vector, under which the identity $-G_A/G = (\mu_0^2/m_\pi^2)\sqrt{Z_3}(f_1/f_0)\, d_\pi(0)\, F_\pi(0)$ holds, is somewhat weakened. It is also demonstrated that the « generalized Ward identity » derived by Bernstein *et al.* should really be replaced by the equation (2.20) in Section 2. The pion decay process into leptons is discussed by introducing the direct coupling between them, in addition to the four-fermion interaction. It is pointed out that the presence of the direct coupling between pion and leptons does not alter the above relation for the ratio G_A/G, its value being independent also of the constant for direct coupling.

1. – Introduction.

Gell-Mann–Lévy have proposed models of the axial vector interaction in which the axial vector current satisfies

$$P^{\lambda}_{\mu,\mu}(x) = ia\pi^{\alpha}(x)\,, \tag{1.1}$$

where $\pi^{\lambda}(x)$ is the pion field [1]. These models have been elaborated by several other authors [2,3].

We propose to examine in this paper the necessity of the condition (1.1), by taking a special model described by the Lagrangian (2.1), where the con-

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[1] M. Gell-Mann and M. Lévy: *Nuovo Cimento*, **16**, 705 (1960).

[2] J. Bernstein, M. Gell-Mann and L. Michel: *Nuovo Cimento*, **16**, 560 (1960).

[3] J. Bernstein, S. Fubini, M. Gell-Mann and W. Thirring: *Nuovo Cimento*, **17**, 757 (1960).

dition (1.1) is somewhat relaxed. We shall show that under the relaxed condition

$$P^{\alpha}_{\mu,\mu}(x) = i(a + b\Box)\pi^{\alpha}(x) , \tag{1.2}$$

we still arrive at the relation

$$-G_A/G = \frac{\mu_0^2}{m_\pi^2}\sqrt{Z_3}\frac{f_1}{f_0}d_\pi(0)F_\pi(0) , \tag{1.3}$$

which was obtained by GELL-MANN–LÉVY under (1.1) [1] (*). We also point out that the « generalized Ward identity » derived by BERNSTEIN *et al.* should really be replaced by the equation (2.20) in Section 2 [2]. We then apply the relation (1.2) to the π-decay process. It is however remarked that nothing definite can be concluded from this kind of model, in spite of Gell-Mann–Lévy's attempt, since this model corresponds to the introduction of direct coupling terms between a pion and its decay particles. An important problem is rather to obtain the observed value $-G_A/G = 1.25$ from the renormalization theory of strong coupling. However, we will not attempt to tackle this problem, but will illustrate a method of deriving the identity between strong and weak interactions, similar to that used to prove the generalized Ward identity in quantum electrodynamics [4]. Two examples are given in the Appendix.

2. – Gell-Mann-Lévy's identity.

We shall discuss a system described by the following Lagrangian density:

$$\mathscr{L}(x) = \mathscr{L}^S(x) + \mathscr{L}^W(x) , \tag{2.1}$$

$$\mathscr{L}^S(x) = -Z_2\bar{N}(x)[\gamma\partial + m]N(x) + \delta m Z_2\bar{N}(x)N(x) - \tfrac{1}{2}Z_3\{\pi^{\alpha}_{,\mu}(x)\pi^{\alpha}_{,\mu}(x) + m_\pi^2\pi^{\alpha}(x)\pi^{\alpha}(x)\} + \tfrac{1}{2}Z_3\delta m_\pi^2\pi^{\alpha}(x)\pi^{\alpha}(x) - if_1Z_1\bar{N}(x)\gamma_5\gamma_\mu\tau^{\alpha}N(x)\pi^{\alpha}_{,\mu}(x) , \tag{2.2}$$

$$\mathscr{L}^W(x) = -\frac{1}{4}\frac{1}{m_B^4}F^{\alpha}_{\mu\nu}(x)F^{\alpha}_{\mu\nu}(x) - \frac{1}{2}\frac{1}{m_B^2}B^{\alpha}_{\mu}(x)B^{\alpha}_{\mu}(x) - iG_Az\bar{N}(x)\gamma_5\gamma_\mu\tau^{\alpha}N(x)B^{\alpha}_{\mu}(x) - G_Az\beta\pi^{\alpha}_{,\mu}(x)B^{\alpha}_{\mu}(x) , \tag{2.3}$$

where

$$F^{\alpha}_{\mu\nu}(x) = B^{\alpha}_{\nu,\mu}(x) - B^{\alpha}_{\mu,\nu}(x)$$

(*) The notations used here are the same as Gell-Mann–Lévy's [1].

[4] Y. TAKAHASHI: *Nuovo Cimento*, **6**, 371 (1957). See also K. NISHIJIMA: *Phys. Rev.*, **119**, 485 (1960); K. SYMANZIK: *Nuovo Cimento*, **11**, 269 (1959).

and β is a constant which is unrestricted throughout our discussion. A quantity β' defined by

$$\beta' \equiv \beta f_1 Z_2 Z_3^{-1} \tag{2.4}$$

will often be used instead of β. In equations (2.2) and (2.3), we have used « renormalized » quantities, but of course this does not mean that all the divergences are eliminated. The B_μ^α-field is normalized so that $[G_A] = L^2$ in the units $\hbar = c = 1$.

We now examine the connection between the strong vertex part with the coupling constant f_1 and the weak vertex part with the coupling constant G_A, which is assumed to be small, and is taken into account only up to the first order in G_A.

The equations of motion for the respective fields are

$$(\gamma\partial + m)N(x) = \delta m\, N(x) - if_1 Z_1 Z_2^{-1} \gamma_5 \gamma_\mu \tau^\chi N(x) \pi^\alpha_{,\mu}(x) - \\ - iG_A z Z_2^{-1} \gamma_5 \gamma_\mu \tau^\chi N(x) B^\chi_\mu(x)\,, \tag{2.5}$$

$$(\Box - m_\pi^2)\pi^\alpha(x) = -\delta m_\pi^2 \pi^\chi(x) - if_1 Z_1 Z_3^{-1} [\bar N(x)\gamma_5\gamma_\mu \tau^\alpha N(x)]_{,\mu} - \\ - \beta G_A z Z_3^{-1} B^\chi_{\mu,\mu}(x)\,, \tag{2.6}$$

$$[(\Box - m_B^2)\delta_{\mu\nu} - \partial_\mu\partial_\nu] B^\alpha_\nu(x) = iG_A z m_B^4 \bar N(x)\gamma_5\gamma_\mu\tau^\alpha N(x) + G_A z m_B^4 \beta \pi^\alpha_{,\mu}(x)\,. \tag{2.7}$$

Since we can easily establish

$$B^\alpha_{\mu,\mu}(x) = -G_A z m_B^2 \beta \mu_0^2 \pi^\alpha(x) - iG_A z m_B^2 (1-\beta')[\bar N(x)\gamma_5\gamma_\mu\tau^\alpha N(x)]_{,\mu}\,, \tag{2.8}$$

with

$$\mu_0^2 = m_\pi^2 - \delta m_\pi^2\,, \tag{2.9}$$

we obtain

$$(\Box - \mu_0^2)\pi^\alpha(x) = -if_1 Z_1 Z_3^{-1}[\bar N(x)\gamma_5\gamma_\mu\tau^\chi N(x)]_{,\mu} \tag{2.10}$$

within the approximation adopted here.

The quantization can be carried out in the standard fashion. Particularly, we have, at $x_0 = y_0$,

$$\{N(x), \bar N(y)\} = Z_2^{-1}\gamma_4 \delta^3(x-y)\,, \tag{2.11}$$

$$[N(x), B_4^\chi(y)] = iG_A z Z_2^{-1} m_B^2 (1-\beta')\gamma_5\tau^\alpha N(y)\,\delta^3(x-y)\,, \tag{2.12}$$

$$[N(x), \dot\pi^\chi(y)] = -f_1 Z_1 Z_2^{-1} Z_3^{-1}\gamma_5\tau^\alpha N(y)\,\delta^3(x-y)\,. \tag{2.13}$$

We obtain therefore

$$(2.14)\quad (\Box_z - \mu_0^2)\ T[N(x), \bar{N}(y), \pi^\alpha(z)]_0 -$$
$$- f_1 Z_1 Z_3^{-1} \{ i\langle T[N(x), \bar{N}(y), \{\bar{N}(z)\gamma_5\gamma_\mu\tau^\alpha N(z)\}_{,\mu}]\rangle_0 +$$
$$+ Z_2^{-1}\delta^4(x-z)\gamma_5\tau^\alpha\langle T[N(z), \bar{N}(y)]\rangle_0 + Z_2^{-1}\langle T[N(x), \bar{N}(z)]\rangle_0\gamma_5\tau^\alpha\delta^4(z-y)\} =$$
$$= -if_1 Z_1 Z_3^{-1}\frac{\partial}{\partial z_\mu}\langle T[N(x), \bar{N}(y), \bar{N}(z)\gamma_5\gamma_\mu\tau^\alpha N(z)]\rangle_0 ,$$

and

$$(2.15)\quad \frac{\partial}{\partial z_\mu}\langle T[N(x), \bar{N}(y), B_\mu^\alpha(z)]\rangle_0 = -G_A z m_B^2 \mu_0^2 \beta\langle T[N(x), \bar{N}(y), \pi^\alpha(z)]\rangle_0 -$$
$$- G_A z m_B^2 (1-\beta')\{ i\langle T[N(x), \bar{N}(y), \{\bar{N}(z)\gamma_5\gamma_\mu\tau^\alpha N(z)\}_{,\mu}]\rangle_0 +$$
$$+ Z_2^{-1}\delta^4(x-z)\gamma_5\tau^\alpha\langle T[N(z), \bar{N}(y)]\rangle_0 + Z_2^{-1}\langle T[N(x), \bar{N}(z)]\rangle_0\gamma_5\tau^\alpha\delta^4(z-y)\} =$$
$$= -G_A z m_B^2 \mu_0^2\beta\langle T[N(x), \bar{N}(y), \pi^\alpha(z)]\rangle_0 -$$
$$- iG_A z m_B^2 (1-\beta')\frac{\partial}{\partial z_\mu}\langle T[N(x), \bar{N}(y), \bar{N}(z)\gamma_5\gamma_\mu\tau^\alpha N(z)]\rangle_0 ,$$

using equations (2.8), (2.10), (2.11), (2.12) and (2.13).

Eliminating the last terms of (2.14) and (2.15), we arrive at

$$(2.16)\quad \frac{\partial}{\partial z_\mu}\langle T[N(x), \bar{N}(y), B_\mu^\alpha(z)]\rangle_0 = -G_A z m_B^2 \mu_0^2\beta\langle T[N(x), \bar{N}(y), \pi^\alpha(z)]\rangle_0 +$$
$$+ \frac{G_A z m_B (1-\beta')}{f_1 Z_1 Z_3^{-1}}(\Box_z - \mu_0^2)\langle T[N(x), \bar{N}(y), \pi^\alpha(z)]\rangle_0 ,$$

which will provide us with the required relation between strong and weak vertex parts, as follows.

We first define strong and weak vertex parts, respectively, by

$$(2.17)\quad \langle T[N(x), \bar{N}(y), \pi^\alpha(z)]\rangle_0 =$$
$$= f_1\int \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'\, S'(x-x')\,\Gamma_{5\nu}^\alpha(x'-z';\, z'-y')\,S'(y'-y)\,\frac{\partial}{\partial z_\nu'}\,\Delta'(z'-z) ,$$

and

$$(2.18)\quad \langle T[N(x), \bar{N}(y), B_\mu^\alpha(z)]\rangle_0 =$$
$$= G_A\int \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'\, S'(x-x')\,\Lambda_{5\nu}^\alpha(x'-z';\, z'-y')\,S'(y'-y)\,\Lambda_{\nu\mu}(z'-z) ,$$

where

$$
(2.19)\qquad \begin{cases} \langle T[N(x), \bar{N}(y)]\rangle_0 = S'(x-y)\,, \\ \langle T[\pi^\alpha(x), \pi^\beta(y)]\rangle_0 = \delta^{\alpha\beta}\Delta'(x-y)\,, \\ \langle T[B^\alpha_\mu(x), B^\beta_\nu(y)]\rangle_0 = m_B^4\delta^{\alpha\beta}\left\{\delta_{\mu\nu} - \frac{1}{m_B^2}\partial_\mu\partial_\nu\right\}\Delta_c(x-y)\,, \\ (\Box - m_B^2)\,\Delta_c(x-y) = i\,\delta^4(x-y)\,. \end{cases}
$$

Substituting (2.17) and (2.18) into (2.16), and transforming into momentum space, we have

$$
(2.20)\qquad i(p-q)_\nu A^\alpha_{5\nu}(p;q) = i(p-q)_\nu z Z_1^{-1} Z_3 \mu_0^2 \beta' \Gamma^\alpha_{5\nu}(p;q)\,\Delta'[(p-q)^2] + \\ + i(p-q)_\nu z Z_1^{-1} Z_3 (1-\beta')\,\Gamma^\alpha_{5\nu}(p;q)[(p-q)^2 + \mu_0^2]\,\Delta'[(p-q)^2]\,.
$$

As is seen in (2.20), the relation between weak $A^\alpha_{5\nu}(p;q)$ and strong $\Gamma^\alpha_{5\nu}(p;q)$ vertices depends on the parameter β'.

In order to derive the $-G_A/G$ formula (1.3), we proceed as follows:

We put

$$
(2.21)\qquad A^\alpha_{5\nu}(p;q) \equiv A^\alpha_{5\nu}(p;q) - z Z_1^{-1} Z_3[\mu_0^2 + (1-\beta')(p-q)^2]\Delta'[(p-q)^2]\Gamma^\alpha_{5\nu}(p;q) = \\ = \tau^\alpha\gamma_5\gamma_\nu\{A_1(p;q) - z Z_1^{-1} Z_3[\mu_0^2 + (1-\beta')(p-q)^2]\Delta'[(p-q)^2]\,F_\pi(p;q)\} + \\ + \tau^\alpha\gamma_5 p_\nu\{\ldots\} + \tau^\alpha\gamma_5 q_\nu\{\ldots\} + \tau^\alpha\gamma_5\sigma_{\nu\mu}p_\mu\{\ldots\} + \tau^\alpha\gamma_5\sigma_{\nu\mu}q_\mu\{\ldots\}\,,
$$

where the terms in the curly brackets do not contain any γ-matrices, and are functions of $-p^2$, q^2, $(p-q)^2$ and sign functions of p_0, q_0 and $p_0 - q_0$ only; in particular, we renormalize so that

$$
(2.22)\qquad \begin{cases} A_1(p;q)_{\substack{-p^2=m^2 \\ -q^2=m^2 \\ -(p-q)^2=0}} = 1\,, \\ F_\pi(p;q)_{\substack{-p^2=m^2 \\ -q^2=m^2 \\ -(p-q)^2=m_\pi^2}} = 1\,. \end{cases} \qquad (p_0, q_0 > 0)\,.
$$

The relation (2.20) then implies

$$
(2.23)\qquad i(p-q)_\nu A^\alpha_{5\nu}(p;q) = 0\,,
$$

which reduces to

$$
(2.24)\qquad A^\alpha_{5\nu}(p;p) = 0\,.
$$

Multiplying by $\tau^\alpha\gamma_5\gamma_\nu$ and taking the trace, we obtain

$$
(2.25)\qquad A_1(p;p) = z Z_1^{-1} Z_3 \mu_0^2 \Delta'(0)\, F_\pi(p;p)
$$

which yields at $-p^2 = m^2$ $(p_0 > 0)$,

$$1 = zZ_1^{-1}Z_3\mu_0^2 \frac{1}{m_\pi^2} d_\pi(0) F_\pi(0) \quad (*) \tag{2.26}$$

where

$$\begin{cases} F_\pi(0) \equiv F_\pi(p\,;\,p)_{-p^2=m^2\,(p_0>0)}\,, \\ d_\pi(0) \equiv m_\pi^2 \Delta'(0)\,. \end{cases} \tag{2.27}$$

Remembering

$$\begin{cases} G_A = -Gz^{-1}Z_2\,, \\ f_1 = f_0 Z_1^{-1} Z_2 Z_3^{\frac{1}{2}}\,, \end{cases} \tag{2.28}$$

we arrive at

$$-G_A/G = \frac{\mu_0^2}{m_\pi^2}\sqrt{Z_3}\frac{f_1}{f_0} d_\pi(0) F_\pi(0)\,. \tag{2.29}$$

Notice that the relation (2.29) holds independently of the value of β'.

3. – Application to π-μ-decay.

We shall now apply the argument in the previous section to the π-μ-decay process. Limiting ourselves to the first order in G_A, we can replace the vector $(B_\mu^1 - iB_\mu^2)$ by $-i\sqrt{2}[\overline{\nu}(x)\gamma_\mu(1+\gamma_5)\mu(x)]$ (**). Then, following Gell-Mann–Lévy's method [1], the rate of π-μ-decay is given as:

$$\Gamma_\pi = m_\pi \varphi_{\beta'}^{-2}(0)[(1.56 \pm 0.2)\cdot 10^{-16}]\,, \tag{3.1}$$

where

$$\varphi_{\beta'}(0) = d_\pi(0) F_\pi(0) \frac{\mu_0^2}{\mu_0^2 + m_\pi^2(\beta' - 1)}\,. \quad (***) \tag{3.2}$$

(*) $\mu_0^2 \neq 0$ is assumed.

(**) The B_μ^α-field can also be considered as an intermediate boson.

(***) $$\varphi_{\beta'}(k^2) = \frac{\mu_0^2 + (1-\beta')k^2}{\mu_0^2 + (\beta'-1)m_\pi^2} d_\pi(k^2) F_\pi(k^2),$$

and

$$\varphi_{\beta'}(-m_\pi^2) = 1\,,$$

since

$$d_\pi(-m_\pi^2) = 1\,, \qquad F_\pi(-m_\pi^2) = 1\,.$$

Special values $\beta'=1$ and $\beta'=0$, give, respectively

$$\begin{cases} \varphi_1(0) = d_\pi(0) F_\pi(0) , \\ \varphi_0(0) = d_\pi(0) F_\pi(0) \dfrac{\mu_0^2}{\mu_0^2 - m_\pi^2} . \end{cases} \tag{3.3}$$

The case $\beta'=1$ was treated by GELL-MANN and LÉVY, who expected that $\varphi_\pi(0)$ is not far from unity, in order to explain the observed rate of π-μ-decay

$$\Gamma_\pi = m_\pi [1.84 \pm 0.04) \cdot 10^{-16}] . \tag{3.4}$$

It is, however, now clear that, since $d_\pi(0)F_\pi(0)$ and μ_0^2 are independent of β', one can choose β' in such a way that equation (3.1) gives (3.4), that is,

$$\beta' = 1 - \frac{\mu_0^2}{m_\pi^2} [1 - 1.08\, d_\pi(0) F_\pi(0)] . \tag{3.5}$$

Thus, we can obtain good agreement with the π-μ-decay experiment by choosing β' properly. However, a question still remains as to what value we can get for $-G_A/G$, which is given by equation (2.22). We must be able to explain it in terms of the strong interaction only, as long as we persist with the view that it is really a renormalization effect. Its value is clearly independent of β'.

* * *

The authors thank Mr. C. RYAN for his valuable discussion.

APPENDIX

We shall obtain other identities which hold between strong and weak interactions.

I. *Relation between strong PV and weak PS coupling.* – The total Lagrangian $\mathscr{L}$ is composed of two parts

$$\mathscr{L} = \mathscr{L}^S + \mathscr{L}^W ,$$

where

$$\mathscr{L}^S = - Z_2 \bar{\psi}(x)(\gamma^\mu \partial_\mu + M)\psi(x) + \delta M Z_2 \bar{\psi}(x)\psi(x) - \tfrac{1}{2} Z_3 [\varphi_{,\mu}(x)\varphi_{,\mu}(x) + m^2 \varphi(x)\varphi(x)] + \tfrac{1}{2}\delta m^2 Z_3 \varphi(x)\varphi(x) - f_1 Z_1 \bar{\psi}(x)\gamma_5\gamma_\mu \psi(x)\varphi_{,\mu}(x) , \tag{A.1}$$

$$\mathscr{L}^W = - \tfrac{1}{2}[B_{,\mu}(x) B_{,\mu}(x) + \varkappa^2 B(x) B(x)] - i G_1 \varkappa \bar{\psi}(x)\gamma_5 \psi(x) B(x) . \tag{A.2}$$

The equations of motion and the commutation relations are

$$\text{(A.3)}\quad \begin{cases} (\gamma\partial + M)\psi(x) = -if_1 Z_1 Z_2^{-1}\gamma_5\gamma_\mu \psi(x)\varphi_\mu(x) - iG_1 z Z_2^{-1}\gamma_5\psi(x)\ B(x)\,, \\ (\Box - m^2)\varphi(x) = -if_1 Z_1 Z_3^{-1}[\bar\psi(x)\gamma_5\gamma_\mu\psi(x)]_{,\mu} - \delta m^2\varphi(x)\,, \\ (\Box - \varkappa^2)B(x) = iG_1 z\,\bar\psi(x)\gamma_5\psi(x)\,, \end{cases}$$

$$\text{(A.4)}\quad \begin{cases} \{\psi(x), \bar\psi(x)\} = Z_2^{-1}\gamma_4\,\delta^3(x-y)\,, \\ [\dot\varphi(x), \varphi(y)] = -iZ_3^{-1}\,\delta^3(x-y)\,, \\ [\dot\varphi(x), \psi(x)] = f_1 Z_1 Z_2^{-1} Z_3^{-1}\gamma_5\psi(y)\,\delta^3(x-y)\,. \\ \qquad\qquad (x_0 = y_0)\,. \end{cases}$$

We can also easily verify

$$\text{(A.5)}\quad [\bar\psi(x)\gamma_5\gamma_\mu\psi(x)]_{,\mu} = -2M_0\,\bar\psi(x)\gamma_5\psi(x) - 2iG_1 z Z_2^{-1}\,\bar\psi(x)\psi(x)B(x)\,.$$

By using (A.3), (A.4), and (A.5), we have

$$\begin{aligned}\text{(A.6)}\quad & (\Box_z - m_1^2)\langle T[\psi(x), \bar\psi(y), \varphi(z)]\rangle_0 = \\ & = -f_1 Z_1 Z_2^{-1} Z_3^{-1}\{\delta^4(x-z)\gamma_5\, T[\psi(z), \psi(y)]_0 + T[\psi(x), \psi(z)]_0\gamma_5\,\delta^4(z-y)\} + \\ & + 2if_1 Z_1 Z_3^{-1} M_0\langle T[\psi(x), \bar\psi(y), \bar\psi(z)\gamma_5\psi(z)]\rangle_0 + \\ & + 2f_1 G_1 z Z_1 Z_2^{-1} Z_3^{-1}\langle T[\psi(x), \bar\psi(y), \bar\psi(z)\psi(z)B(z)]\rangle_0 \quad (*).\end{aligned}$$

On the other hand, we can readily prove

$$\text{(A.7)}\quad (\Box_z - \varkappa^2)\langle T[\psi(x), \bar\psi(y), B(z)]\rangle_0 = iG_1 z\langle T[\psi(x), \bar\psi(y), \bar\psi(z)\gamma_5\psi(z)]\rangle_0\,.$$

Neglecting the last term in (A.6), which is of the order $O(G_1^2)$, we have from (A.6) and (A.7),

$$\begin{aligned}\text{(A.8)}\quad & (\Box_z - m_0^2)\langle T[\psi(x), \bar\psi(y), \varphi(z)]\rangle_0 = -f_1 Z_1 Z_2^{-1} Z_3^{-1}\{\delta^4(x-z)\gamma_5\langle T[\psi(z), \bar\psi(y)]\rangle_0 + \\ & + \langle T[\psi(x), \bar\psi(z)]\rangle_0\gamma_5\,\delta^4(z-y)\} + \frac{2f_1 Z_1 Z_3^{-1}}{G_1 z} M_0(\Box_z - \varkappa^2)\langle T[\psi(x), \bar\psi(y), B(z)]\rangle_0\,.\end{aligned}$$

This is, in momentum space,

$$\begin{aligned}\text{(A.9)}\quad & i(p-q)_\nu S'(p)\,\Gamma_{5\nu}(p;q)\,S'(q)[(p-q)^2 + m_0^2]\,\Delta'[(p-q)^2] = \\ & = -Z_1 Z_2^{-1} Z_3^{-1}[S'(p)\gamma_5 + \gamma_5 S'(q)] + \frac{2M_0 Z_1 Z_3^{-1}}{z} S'(p)\,\Lambda_5(p;q)\,S'(q)\,,\end{aligned}$$

(*) $m_0^2 = m^2 - \delta m^2$, $M_0 = M - \delta M$.

where the vertex parts $\Gamma_{5\nu}$ and Λ_5 are defined in the same way as (2.17) and (2.18).

II. *Relation between strong PS and weak PV coupling.* – Take the Lagrangian $\mathscr{L} = \mathscr{L}^S + \mathscr{L}^W$ with

$$\mathscr{L}^S = -Z_2\,\bar\psi(x)(\gamma\partial + M)\psi(x) + \delta M Z_2\,\bar\psi(x)\psi(x) - \tag{A.10}$$
$$-\tfrac{1}{2}Z_3[\varphi_{,\mu}(x)\varphi_{,\mu}(x) + m^2\varphi(x)\varphi(x)] + \tfrac{1}{2}\,\delta m^2 Z_3\varphi(x)\varphi(x) - ig_1 Z_1\,\bar\psi(x)\gamma_5\psi(x)\varphi(x)\,,$$

$$\mathscr{L}^W = -\tfrac{1}{2}[B_{,\mu}(x)B_{,\mu}(x) + \varkappa^2 B(x)B(x)] - iF_1 z\,\bar\psi(x)\gamma_5\gamma_\mu\psi(x)B_{,\mu}(x) + \tag{A.11}$$
$$+ 2F_1 z g_1 Z_1 Z_2^{-1}\,\bar\psi(x)\psi(x)\varphi(x)B(x)\,.$$

The last term of (A.11) is added so that a simple relation follows, as is seen below.

In the approximation considered here, that is, $O(F_1^2)$ is neglected, we have

$$(\Box_z - \varkappa^2)\langle T[\psi(x), \bar\psi(y), B(z)]\rangle_0 = \tag{A.12}$$
$$= -F_1 z Z_2^{-1}\{\delta^4(x-z)\gamma_5\langle T[\psi(z), \bar\psi(y)]\rangle_0 + \langle T[\psi(x), \bar\psi(z)]\rangle_0\gamma_5\,\delta^4(z-y)\} +$$
$$+ 2\,\frac{F_1 z M_0}{g_1 Z_1 Z_3^{-1}}\,(\Box_z - m_0^2)\langle T[\psi(x), \bar\psi(y), \varphi(z)]\rangle_0\,.$$

In momentum space, (A.12) is

$$i(p-q)_\nu S'(p)\,\Lambda_{5\nu}(p;q)S'(q) = \tag{A.13}$$
$$= 2\,\frac{zM_0}{Z_1 Z_3^{-1}}\,S'(p)\,\Gamma_5(p;q)\,S'(q)\{(p-q)^2 + m_0^2\}\Delta'[(p-q)^2] - zZ_2^{-1}[\gamma_5 S'(q) + S'(p)\gamma_5]\,.$$

RIASSUNTO (*)

Si analizzano ulteriormente i modelli della interazione debole del vettore assiale proposti da Gell-Mann e Lévy. Si rende un po' meno stringente la condizione $P^\alpha_{\mu,\mu}(x) = ia\pi^\alpha(x)$, per il vettore assiale debole, per il quale vale l'identità $-G_A/G = (\mu_0^2/m_\pi^2)\cdot\sqrt{Z}(f_1/f_0)d_\pi(0)F_\pi(0)$. Si dimostra anche che la « identità generalizzata di Ward » derivata da Bernstein *et al.* deve essere effettivamente sostituita dall'equazione (2.20) della Sezione **2**. Si discute il processo di decadimento del pione in leptoni, introducendo fra loro l'accoppiamento diretto, in aggiunta all'interazione a quattro fermioni. Si mette in evidenza che la presenza dell'accoppiamento diretto tra pioni e leptoni non altera la suddetta relazione per il rapporto G_A/G, il cui valore è indipendente anche dalla costante di accoppiamento diretto.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XIX, N. 3 1° Febbraio 1961

Effect of Radiofrequency Resonance on the Natural Line Form (*).

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(ricevuto l'11 Ottobre 1960)

Summary. — The form of Zeeman lines in the presence of a resonant rotating r.f. magnetic field is determined. The spontaneous emission distributions are obtained from the steady-state solutions, leading to a prediction of the splitting of Zeeman lines at rotation frequencies corresponding to well separated single- and multiple-quantum resonance frequencies, as well as at the resonance frequency in the Majorana case.

1. – Introduction.

Some time ago Bitter (1) and Pryce (2) indicated the possibility of a new method of detecting radiofrequency resonance, by means of certain changes in the radiation emitted by an atom when it is subjected to a radiofrequency field at a resonance frequency between its Zeeman levels in a constant magnetic field. At that time the observation of such an effect appeared difficult to perform, since it would require large radiofrequency field amplitudes and high resolution of the emitted radiation in order to detect the changes produced by applying the radiofrequency field. However, the possibility of an experiment on the nuclear Zeeman lines by means of the Mössbauer effect has renewed interest in this problem (3).

(*) Work performed under the auspices of the U. S. Atomic Energy Commission.

(1) F. Bitter: *Phys. Rev.*, **76**, 833 (1949).

(2) M. H. L. Pryce: *Phys. Rev.*, **77**, 136 (1950). We are indebted to Professor A. Abragam for bringing references (1) and (2) to our attention when we informed him of our results.

(3) A preliminary experimental attempt was reported by E. C. Avery, C. Littlejohn, G. J. Perlow and B. Smaller at the Allerton Park Conference, University of Illinois (June 1960).

In the present note we study the influence of radiofrequency resonance on the natural line form of the emitted radiation. The general theory is closely related to the steady-state solutions found in the study of multiple-quantum transitions [4].

2. – The two-level case.

We first consider a system undergoing radiative decay from a pair of levels α and β to levels α' and β' respectively, in the presence of a rotating r.f. field which produces transitions between α and β (Fig. 1). We wish to study the effect on the emitted spectral lines when the angular frequency of rotation ω is close to the resonance frequency $\omega_{\alpha\beta} = (E_\alpha - E_\beta)/\hbar$ between the upper states, and the amplitude of the rotating field is sufficiently large to produce an appreciable probability of transfer between α and β during their mean life-time. We shall assume that both states decay at the same rate.

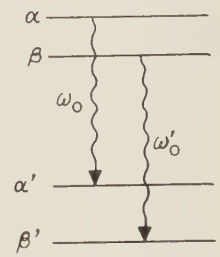

Fig. 1. – Decay transitions in the presence of a r.f. field.

For the present we neglect the influence of other levels on the resonance $\alpha \leftrightarrow \beta$. In the following section the possibility of transitions to other levels will be taken into account, leading to corresponding results also at the resonance frequencies associated with well separated single- and multiple-quantum transitions, as well as at the single resonance frequency in the case of equally spaced levels.

To investigate the effect of the radiofrequency resonance on the natural line form, we study the solution of the system of equations

$$(1a) \qquad i\dot{b}_{\alpha 0} = H \exp[i(\omega_{\alpha\beta} - \omega)t] b_{\beta 0} + \sum_\lambda H_{\alpha 0|\alpha' 1_\lambda} \exp[i(\omega_{\alpha\alpha'} - \omega_\lambda)t] b_{\alpha' 1} \,,$$

$$(1b) \qquad i\dot{b}_{\alpha' 1_\lambda} = H_{\alpha' 1_\lambda|\alpha 0} \exp[i(\omega_\lambda - \omega_{\alpha\lambda'})t] b_{\alpha 0} \,,$$

$$(1c) \qquad i\dot{b}_{\beta 0} = H \exp[i(\omega - \omega_{\alpha\beta})t] b_{\alpha 0} + \sum_\sigma H_{\beta 0|\beta' 1_\sigma} \exp[i(\omega_{\beta\beta'} - \omega_\sigma)t] b_{\beta' 1} \,,$$

$$(1d) \qquad i\dot{b}_{\beta' 1_\sigma} = H_{\beta' 1_\sigma|\beta 0} \exp[i(\omega_\sigma - \omega_{\beta\beta'})t] b_{\beta 0} \,,$$

subject to the condition that initially no photons are in the radiation field,

$$(2) \qquad b_{\alpha' 1_\lambda}(0) = b_{\beta' 1_\sigma}(0) = 0 \,.$$

[4] M. N. Hack: *Phys. Rev.*, **100**, 975 (A) (1955); **104**, 84 (1956); H. Salwen: *Phys. Rev.*, **99**, 1274 (1955); C. Besset, J. Horowitz, A. Messiah and J. M. Winter: *Journ. Phys. et Rad.*, **15**, 251 (1954); J. M. Winter: *Ann. Phys.*, **4**, 745 (1959).

For $H = 0$, *i.e.*, for zero amplitude of the radiofrequency field, these equations reduce to the equations of Weisskopf and Wigner (5) for describing the radiative decay of excited states. For vanishing matrix elements of the interaction with the radiation field, $H_{\alpha 0|\alpha' 1_\lambda} = 0$, etc., they would reduce to the equations describing the transitions produced by the radiofrequency field alone.

In general eqs. (1) are too complicated to solve exactly. Nevertheless, for the cases of practical interest a satisfactory approximate solution can be obtained. For this purpose we look for solutions of the form

$$b_{\alpha 0} = C_\alpha \exp[-ipt], \tag{3a}$$

$$b_{\beta 0} = C_\beta \exp[-i(p + \omega_{\alpha\beta} - \omega)t]. \tag{3b}$$

Substituting (3*a*) into (1*b*) and integrating, subject to the initial condition (2), and inserting the result

$$b_{\alpha'} = H_{\alpha'\lambda} C_\lambda \frac{\exp[-i(\omega_{\alpha\lambda'} + p - \omega_\lambda)t] - 1}{\omega_{\lambda\alpha'} + p - \omega_\lambda}, \tag{4a}$$

and eqs. (3) into (1*a*) gives

$$pC_\lambda = HC_\beta + C_\alpha \sum_\lambda |H_{\lambda\lambda'}|^2 \frac{1 - \exp[i(\omega_{\alpha\alpha'} + p - \omega_\lambda)t]}{\omega_{\alpha\alpha'} + p - \omega_\lambda}. \tag{5}$$

(For simplicity photon occupation numbers have been suppressed.)

It is now important to note that for the times of actual interest, *i.e.*, during a long time interval which includes the mean lifetime of the decaying states, by virtue of the smallness of the decay constant (line width) compared to the line frequency, the last term in eq. (5) is practically constant (6). Denoting

(5) V. Weisskopf and E. Wigner: *Zeits. Phys.*, **63**, 54 (1930).

(6) V. Weisskopf and E. Wigner: ref. (5). One can estimate the upper limit of the interval to be of the order of a few times $\Gamma^{-1} \ln(\omega_0/\Gamma)$, for $\Gamma \ll \omega_0$. At the same time one obtains an estimate for the lower limit (non-linear decay contribution to the change in probability of remaining in the initial state at very early times) as being at most of the order of the period $2\pi/\omega_0$. These results follow already in the Weisskopf-Wigner approximation, even with the usual neglect of the frequency variation of the matrix element and density of states. In this case the second term in the sum in eq. (5) leads to a factor

$$\int_0^\infty \frac{\exp[i(\overline{\omega} - \omega_\lambda - \frac{1}{2}i\Gamma)t]}{\overline{\omega} - \omega_\lambda - \frac{1}{2}i\Gamma} \mathrm{d}\omega_\lambda = 2\pi i + \exp[\tfrac{1}{2}\Gamma t] I,$$

its value by $-(i/2)\Gamma\cdot C_\alpha$ (7), we have

$$\left(p+\frac{i}{2}\Gamma\right)C_\alpha = HC_\beta\,. \tag{6a}$$

In the same way we get

$$b_{\beta'} = H_{\beta'\beta}C_\beta\,\frac{\exp\left[-i(\omega_{\alpha\beta'}+p-\omega-\omega_\sigma)t\right]-1}{\omega_{\alpha\beta'}+p-\omega-\omega_\sigma}\,, \tag{4b}$$

and

$$HC_\alpha = \left(p+\frac{i}{2}\Gamma+\omega_{\alpha\beta}-\omega\right)C_\beta\,, \tag{6b}$$

where by the restriction to the case of equal lifetimes Γ has the same value as in (6*a*).

The simple eigenvalue problem, eqs. (6*a*, *b*), has the solutions

$$p_\pm = \frac{1}{2}(\omega-\omega_{\alpha\beta}) + w_\pm - \frac{i}{2}\Gamma, \tag{7a}$$

where

$$w_\pm = \pm\tfrac{1}{2}\left[(\omega-\omega_{\alpha\beta})^2+4H^2\right]^{\frac{1}{2}}, \tag{7b}$$

and

$$C_{\pm,\alpha} = \pm\frac{H}{|H|}\,\frac{1}{\sqrt{2}}\left[1\pm\frac{\omega_{\alpha\beta}-\omega}{\left[(\omega-\omega_{\alpha\beta})^2+4H^2\right]^{\frac{1}{2}}}\right]^{\frac{1}{2}}, \tag{8a}$$

$$C_{\pm,\beta} = \frac{1}{\sqrt{2}}\left[1\mp\frac{\omega_{\alpha\beta}-\omega}{\left[(\omega-\omega_{\alpha\beta})^2+4H^2\right]^{\frac{1}{2}}}\right]^{\frac{1}{2}}. \tag{8b}$$

where $\Gamma>0$, (7) and the last term furnishes the deviations. For all $t>0$ we have $|\Gamma|\leqslant 1/\overline{\omega}t$, and for $t\gg 1/\overline{\omega}$, $I\sim\{\exp[i\overline{\omega}t]\}/i(\overline{\omega}-i\Gamma/2)t$. More refined estimates taking into account the frequency variation of the matrix element and density of states lead to similar results. The dependence of the decay constant on the r.f. quantities in our case ($\omega \quad \omega_{\alpha\beta}$, H) is also weak since these are very small, of the order of the line width, compared to the frequency of the line. Decay theory has been discussed from a different viewpoint, based on the characteristic functions of the total-energy operator, by L. A. Khalfin: *Sov. Phys. Journ. Exp. Theor. Phys.*, **6**, 1053 (1958).

(7) We suppose the imaginary parts of the decay constants to be absorbed into the energies E_α and $E_{\beta'}$. Then Γ is purely real and moreover, as one readily verifies, $\geqslant 0$. We take Γ strictly positive, since we naturally are not interested in cases where selection rules forbid the decay.

The particular superposition satisfying the initial condition that the system is in the state $\alpha 0$ at time $t = 0$ is given by

$$(9)\qquad \begin{cases} b_{\alpha 0} = C^2_{+\alpha} \exp[-ip_+ t] + C^2_{-\alpha} \exp[-ip_- t], \\ b_{\beta 0} = (C_{+\alpha} C_{+\beta} \exp[-ip_+ t] + C_{-\alpha} C_{-\beta} \exp[-ip_- t]) \exp[-i(\omega_{\alpha\beta} - \omega)t]. \end{cases}$$

The probabilities of finding the system subsequently, at time t, in the states $\alpha 0$ or $\beta 0$ are therefore

$$(10)\qquad \begin{cases} |b_{\alpha 0}(t)|^2 = (1 - P) \exp[-\Gamma t], \\ |b_{\beta 0}(t)|^2 = P \exp[-\Gamma t], \end{cases}$$

where

$$(11)\qquad P = \frac{4H^2}{(\omega - \omega_{\alpha\beta})^2 + 4H^2} \sin^2 \frac{1}{2}[(\omega - \omega_{\alpha\beta})^2 + 4H^2]^{\frac{1}{2}} t,$$

i.e., the Rabi transition probabilities multiplied by the exponential decay factor $\exp[-\Gamma t]$.

The probability amplitudes for finding the system in the lower levels after decay, with a photon of frequency ω_λ or ω_σ respectively in the radiation field, are ([7])

$$(12)\qquad \begin{cases} b_{\alpha'}(\infty) = H_{\alpha'\alpha}\left(\eta_+ C_{+\alpha} \dfrac{1}{\omega_\lambda - \omega_{\alpha\alpha'} - p_+} + \eta_- C_{-\alpha} \dfrac{1}{\omega_\lambda - \omega_{\alpha'} - p_-}\right), \\ b_{\beta'}(\infty) = H_{\beta'\beta}\left(\eta_+ C_{+\beta} \dfrac{1}{\omega_\sigma - \omega_{\alpha\beta'} - p_+ + \omega} + \eta_- C_{-\beta} \dfrac{1}{\omega_\sigma - \omega_{\alpha\beta'} - p_- + \omega}\right), \end{cases}$$

where the constants $\eta_\pm (|\eta_+|^2 + |\eta_-|^2 = 1)$ determine the particular superposition satisfying an arbitrary initial condition.

For the initial condition corresponding to (9), $\eta_+ = C_{+,\alpha}$ and $\eta_- = C_{-,\alpha}$. For

$$(13)\qquad \omega = \omega_{\alpha\beta} \quad \text{(r.f. resonance)},$$

the corresponding emission probabilities are therefore

$$(14)\qquad \begin{cases} |b_{\alpha\to\alpha'}(\infty)|^2_{\text{res}} = |H_{\alpha\alpha'}|^2 \dfrac{(\omega_\lambda - \omega_0)^2 + \Gamma^2/4}{[(\omega_\lambda - \omega_0)^2 - \Gamma^2/4 - H^2]^2 + \Gamma^2(\omega_\lambda - \omega_0)^2}, \\ |b_{\beta\to\beta'}(\infty)|^2_{\text{res}} = |H_{\beta\beta'}|^2 \dfrac{H^2}{[(\omega_\sigma - \omega_0')^2 - \Gamma^2/4 - H^2]^2 + \Gamma^2(\omega_\sigma - \omega_0')^2}, \end{cases}$$

where

$$(15)\qquad \omega_0 = \omega_{\alpha\alpha'}, \qquad \omega_0' = \omega_{\beta\beta'}.$$

Similarly for the condition that the system is initially in the state $\beta 0$ at time $t = 0$,

$$(16)\qquad \begin{cases} |b_{\beta\to\alpha'}(\infty)|^2_{\text{res}} = |H_{\alpha\alpha'}|^2 \dfrac{H^2}{[(\omega_\lambda-\omega_0)^2-\Gamma^2/4-H^2]^2+\Gamma^2(\omega_\lambda-\omega_0)^2}, \\ |b_{\beta\to\beta'}(\infty)|^2_{\text{res}} = |H_{\beta\beta'}|^2 \dfrac{(\omega_\sigma-\omega_0')^2+\Gamma^2/4}{[(\omega_\sigma-\omega_0')^2-\Gamma^2/4-H^2]^2+\Gamma^2(\omega_\sigma-\omega_0')^2}. \end{cases}$$

The effects of the r.f. field on the emitted spectral lines are readily seen in the solutions (14) and (16). They persist and take an even simpler form in the case of equal populations (and random phase difference) in the levels α and β. In this case the emission distribution is the superposition with equal weights of the distributions for the pure steady-state solutions where the components are displaced without change of shape.

Averaging over the inital states and summing over all the final photon states at the energy $h\omega_\lambda$, we obtain for the line $\alpha \to \alpha'$ the photon frequency distribution

$$(17)\qquad P(\omega_\lambda)_{\text{res}} = \frac{\Gamma}{4\pi}\frac{(\omega_\lambda-\omega_0)^2+H^2+\Gamma^2/4}{[(\omega_\lambda-\omega_0)^2-\Gamma^2/4-H^2]^2+\Gamma^2(\omega_\lambda-\omega_0)^2}.$$

The similar distribution for the line $\beta \to \beta'$ is obtained by replacing ω_0 by ω_0'.

The corresponding results in general, *i.e.*, not restricted to r.f. resonance, are

$$(18)\qquad P(\omega_\lambda) = \frac{\Gamma}{4\pi}\cdot$$

$$\frac{\left(\omega_\lambda-\omega_0-\dfrac{\omega-\omega_{\alpha\beta}}{2}\right)^2+\dfrac{\Gamma^2}{4}+\dfrac{1}{4}(\omega-\omega_{\alpha\beta})^2+H^2-(\omega-\omega_{\alpha\beta})\left(\omega_\lambda-\omega_0-\dfrac{\omega-\omega_{\alpha\beta}}{2}\right)}{\left[\left(\omega_\lambda-\omega_0-\dfrac{\omega-\omega_{\alpha\beta}}{2}\right)^2-\dfrac{\Gamma^2}{4}-\dfrac{1}{4}(\omega-\omega_{\alpha\beta})^2-H^2\right]^2+\Gamma^2\left(\omega_\lambda-\omega_0-\dfrac{\omega-\omega_{\alpha\beta}}{2}\right)^2}$$

and the similar relation for the line $\beta \to \beta'$ obtained by replacing ω_0 by $\omega_{\alpha\beta'}-\omega$ and changing the sign of the last term in the numerator.

Fig. 2 shows the behavior of the emission distribution at resonance, for increasing values of the radiofrequency field amplitude. When the field amplitude is sufficiently large that the angular frequency of oscillation between α and β is comparable to the decay rate, the spectral lines $\alpha \to \alpha'$ and $\beta \to \beta'$ each exhibit two peaks. The distribution $P(\omega_\lambda)_{\text{res}}$ is the superposition of two Lorentzian [8] components of equal intensity, and widths equal to Γ, centered at $\omega_0 \pm H$.

[8] The frequency dependence of the factor $\int |H_{\alpha\alpha'}|^2 \varrho(\omega_\lambda)\,d\Omega$ leads to asymmetry of the normal line [5]. For the frequencies of interest, however, because of $\Gamma \ll \omega_0$, we make a negligible error by replacing this factor for simplicity by its value at the

One verifies further that the splitting of the spectral lines appears only for ω close to resonance. As the relative deviation of ω from resonance increases, both lines tend to their natural forms (the intensity of the off-frequency component drops to zero). The emission distribution near resonance in the two-level case is illustrated in Fig. 3.

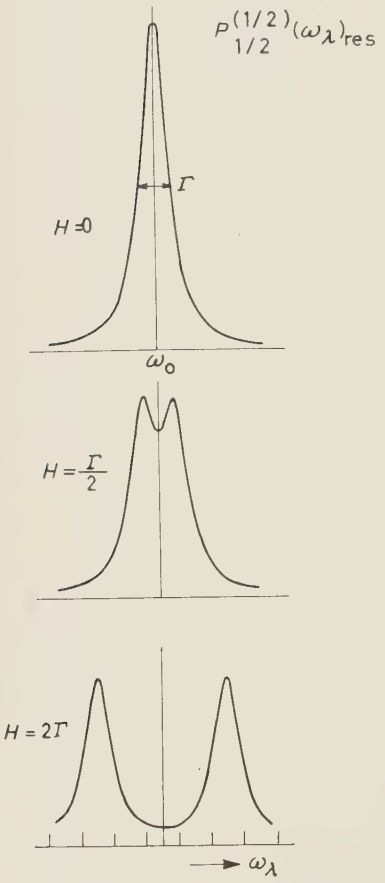

Fig. 2. – Emission distribution $P^{(\frac{1}{2})}_{\frac{1}{2}}(\omega_\lambda)$ at resonance for increasing r.f. amplitude.

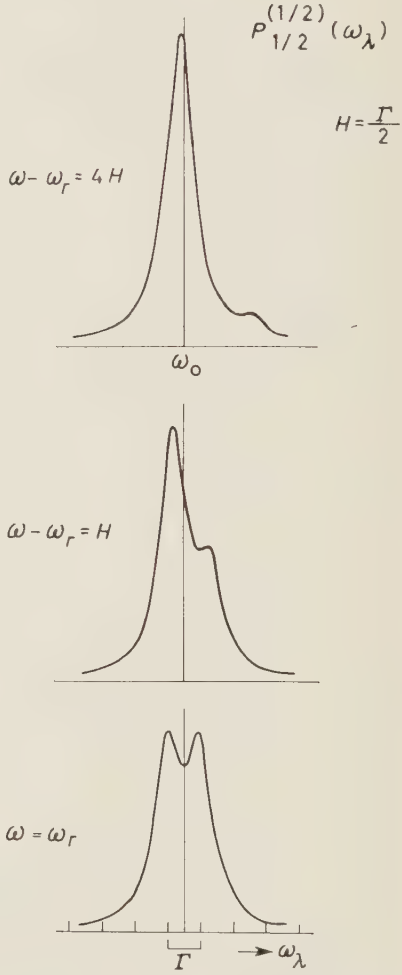

Fig. 3. – Emission distribution $P^{(\frac{1}{2})}_{\frac{1}{2}}(\omega_\lambda)$ for fixed $H=\frac{1}{2}\Gamma$ and varying deviation from resonance.

line center. At large ω_λ it tends to zero by virtue of the retardation, as required for convergence of the total radiated energy. For small ω_λ (frequencies up to and even beyond the line frequency) it has the approximate value $\Gamma\omega_\lambda/\omega_0$:

3. – Arbitrary number of levels.

The preceding treatment of the decay from two levels can readily be extended to the case of an arbitrary number of decaying states (9). We will show that the splitting of spectral lines by r.f. fields also occurs at distinct multiple-quantum resonance frequencies, as well as at the resonance frequency in the case of equally spaced levels.

For this purpose we consider a rotationally invariant system which interacts, through magnetic moments $\gamma_1 \boldsymbol{J}_1$ and $\gamma_2 \boldsymbol{J}_2$, with an external magnetic field consisting of a constant field $\mathscr{H}_0$ and a perpendicular rotating field of constant amplitude $\mathscr{H}$ and angular velocity ω, which causes transitions between the levels of the system in the constant field.

Omitting for the moment the interaction with the radiation field, and choosing the z direction along the constant field and the x axis along the direction of the rotating field at time $t = 0$, we have for the Hamiltonian of the system

$$\hat{H} = H_0 + H' , \tag{19}$$

where

$$H' = (\gamma_1 \boldsymbol{J}_1 + \gamma_2 \boldsymbol{J}_2) \cdot [(\mathscr{H} \cos \omega t)\boldsymbol{i} + (\mathscr{H} \sin \omega t)\boldsymbol{j}] . \tag{20}$$

H_0 contains the interaction with the constant field and the $\boldsymbol{J}_1 \cdot \boldsymbol{J}_2$ interaction, and commutes with the z component of the total angular momentum $J_z = J_{1z} + J_{2z}$.

The Schrödinger equation takes on a more convenient form if we transform to the co-ordinate system of the rotating magnetic field (10) by means of the unitary operator $U = \exp [iJ_z \omega t/\hbar]$. Then

$$\varphi = U\psi = \exp [iJ_z \omega t/\hbar]\psi \tag{21}$$

satisfies

$$i\hbar \frac{\mathrm{d}\varphi}{\mathrm{d}t} = W\varphi , \tag{22}$$

(9) When the calculation is extended in addition to an arbitrary number of lower levels to which a given state can decay, the decay constants combine additively to give the total width (5). It is this summation which leads in practice to the (approximate) equality of lifetimes of the decaying states. When several states decay to the same level, cross terms appear which are rapidly oscillating for $\omega \gg \Gamma$.

(10) I. I. Rabi, N. F. Ramsey and J. Schwinger: *Rev. Mod. Phys.*, **26**, 167 (1954).

where the transformed Hamiltonian

$$W = H_0 - \omega J_z + V \tag{23}$$

with

$$V = \gamma_1 J_{1x} \mathscr{H} + \gamma_2 J_{2x} \mathscr{H} \tag{24}$$

is independent of the time.

The transformed Schrödinger equation (22) has steady-state solutions

$$\varphi = u \cdot \exp[-iwt]\,. \tag{25}$$

By means of this transformation, the problem is reduced to the solution of an eigenvalue problem. As in the two-level case treated in the previous section, the introduction of the decay transitions effectively changes w according to

$$w \to w - \frac{i}{2}\Gamma, \tag{26}$$

where Γ is the decay constant. Here, as before, we consider the case where all levels decay at the same rate. In this case the eigenvalue problem is the same as in the absence of decay. (In the general case, the effect of decay is to introduce constants $\frac{1}{2}i\Gamma_1$, $\frac{1}{2}i\Gamma_2$, ... into the diagonal terms of the eigenvalue equations. In the case treated here, all the Γ's are equal).

For well separated resonance frequencies and for ω in the neighborhood of a resonance frequency, it has been shown ([4]) that the eigenvalue problem can be reduced to a two-level problem, even in the case of the higher order multiple-quantum transitions.

Equations (1) then apply directly, with H replaced by an expression of the type of higher order perturbation theory, ([11]) and $\omega - \omega_{\alpha\beta}$ replaced by a quantity Ω' ([12]). The splitting of the spectral lines thus occurs for each of the resonance frequencies, including the ones corresponding to multiple-quantum transitions. Only the two lines emanating from the particular initial and final levels of the given resonance transition are split by the application of the r.f. field. This follows from the fact that only these levels have appreciable components in the corresponding steady-state solutions which become almost degenerate at the given resonance.

The opposite case of equally spaced levels, where several resonance frequencies coincide, can be treated by similar methods. Apart from the field-

([11]) M. N. HACK: reference ([4]), eq. (51) for δ.
([12]) M. N. HACK: reference ([4]), eq. (42), (50), (51).

independent term of w, the steady-state solutions for this case (Majorana case) can be expressed in the form ([13])

$$C^{(j)}_{\mu m} = d^{(j)}_{\mu m}(\theta)\,, \tag{27}$$

$$w^{(j)}_{\mu} = \mu[(\omega-\omega_r)^2+4H^2]^{\frac{1}{2}}\,, \tag{28}$$

where

$$\cos\theta = \frac{\omega_r-\omega}{[(\omega-\omega_r)^2+4H^2]^{\frac{1}{2}}} \tag{29a}$$

$$\sin\theta = \frac{2H}{[(\omega-\omega_r)^2+4H^2]^{\frac{1}{2}}}\,, \tag{29b}$$

$\omega_r=\gamma_J\mathscr{H}_0$, $2H=\gamma_J\mathscr{H}$, and the $d^{(j)}_{\mu m}$ are representations of the rotation group. Applied to the decay problem, they lead to the probability amplitudes

$$b^{(j)}_{\mu m} = d^{(j)}_{\mu m}\exp\{-i(m[\omega-\omega_r]+\mu[(\omega-\omega_r)^2+4H^2]^{\frac{1}{2}}-\tfrac{1}{2}i\Gamma)t\}\,, \tag{30}$$

$$b'^{(j)}_{\mu m}(\infty) = d^{(j)}_{\mu m}H_{\text{rad}}\frac{1}{\omega_\lambda-(\omega_0+m[\omega-\omega_r]+\mu[(\omega-\omega_r)^2+4H^2]^{\frac{1}{2}})+\frac{1}{2}i\Gamma}\,. \tag{31}$$

In the case of equal populations and random phases, this gives

$$P^{(j)}_m(\omega_\lambda) = \frac{1}{2j+1}\frac{\Gamma}{2\pi}\sum_\mu |d^{(j)}_{\mu m}(\theta)|^2\cdot \cdot\frac{1}{\{\omega_\lambda-(\omega_0+m[\omega-\omega_r]+\mu[(\omega-\omega_r)^2+4H^2]^{\frac{1}{2}})\}^2+\Gamma^2/4}\,, \tag{32}$$

i.e., a superposition of normal components with widths Γ, centers at

$$\omega_0+m[\omega-\omega_r]+\mu[(\omega-\omega_r)^2+4H^2]^{\frac{1}{2}}, \qquad \mu=-j,\ -j+1,\ \ldots j\,,$$

and intensities proportional to the squares of the coefficients of the rotation matrices.

In this case, each line thus splits in general into a number of components equal to the multiplicity $2j+1$. However, in certain cases fewer components may appear, because some of the intensity coefficients vanish. At resonance $(\omega=\omega_r)$, this occurs when the multiplicity is odd, for the central level (the next to last and alternate components are absent) and besides the central

([13]) H. Salwen: reference ([4]), eq. (38), (39).

level for all the alternate levels starting from the next to last (the central component is absent).

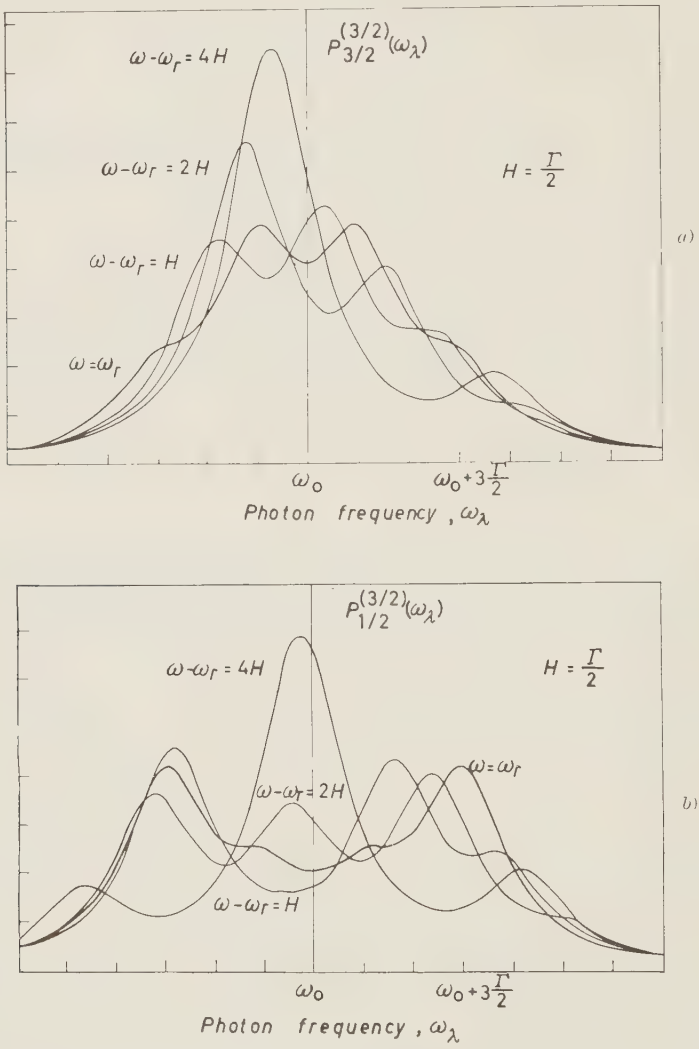

Fig. 4. – Emission distribution $P^{(\frac{3}{2})}_m(\omega_\lambda)$ for fixed $H=\frac{1}{2}\Gamma$ and varying $\omega \geqslant \omega_r$: (*a*) $m=\frac{3}{2}$; (*b*) $m=\frac{1}{2}$. The corresponding curves for $\omega \leqslant \omega_r$ are obtained by reflection in the vertical axis through ω_0. The curves for negative m are obtained by reflection in the same axis and displacement to the corresponding mean emission frequency.

In the particular case of $j=\frac{1}{2}$, eq. (32) reduces to eq. (18) ff. (Figs. 2 and 3). The graphs of $P^{(j)}_m(\omega_\lambda)$ for the case of $j=\frac{3}{2}$ are shown in Fig. 4*a* and *b* for fixed $H=\Gamma/2$.

RIASSUNTO (*)

Si determina la forma delle linee di Zeeman in presenza di un campo magnetico rotante risonante a r.f. Le distribuzioni della emissione spontanea vengono ottenute dalle soluzioni per lo stato continuo, e portano alla predizione della scissione delle linee di Zeeman sia per frequenze di rotazione pari alle ben distinte frequenze di risonanza quantistiche singole e multiple, sia per la frequenza di risonanza nel caso di Majorana.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XIX, N. 3 1° Febbraio 1961

General Relativity without Coordinates.

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(ricevuto il 17 Ottobre 1960)

Summary. — In this paper we develop an approach to the theory of Riemannian manifolds which avoids the use of co-ordinates. Curved spaces are approximated by higher-dimensional analogs of polyhedra. Among the advantages of this procedure we may list the possibility of condensing into a simplified model the essential features of topologies like Wheeler's wormhole and a deeper geometrical insight.

1. – Polyhedra.

In this section we shall first describe our approach for the simple case of 2-dimensional manifold (surfaces). Following Aleksandrov (1) we develop the theory of intrinsic curvature on polyhedra. A general surface is then considered as the limit of a suitable sequence of polyhedra with an increasing number of faces. A rigorous definition of limit is not given here since it would involve a treatment of the topology on the set of all polyhedra and this would carry us too far. It is to be expected however that any surface can be arbitrarily approximated, as closely as wanted, by a suitable polyhedron. The approximation will be bad if we look at the details to the picture but an observer looking at the broad details only will find it quite satisfactory. On any surface we can define an integral Gaussian curvature by carrying out curvature experiments with geodesic triangles.

Let t be one such a triangle and let α, β, γ be its internal angles. If the geometry inside the triangle is not euclidean we have in general $\alpha+\beta+\gamma \neq \pi$.

(*) Nov at the University, Torino.

(1) P. S. Aleksandrov: *Topologia combinatoria* (Torino, 1957).

The Gaussian integral curvature ε_t of t is then defined by $\varepsilon_t = \alpha+\beta+\gamma-\pi$. On a sphere we have $\varepsilon_t = A_t/R^2$, where R is the radius of the sphere and A_t the area of t. If t shrinks to a point P and the limit $\lim_{A\to 0} \varepsilon_t/A_t = K(P)$ exists and is independent of the particular limiting procedure we take $K(P)$ as definition of local Gaussian curvature in P. On a sphere obviously $K(P) = 1/R^2$. It is well known that $\varepsilon_t = \int_t K(P)\,\mathrm{d}A$, where $\mathrm{d}A$ is the area element on the surface and the integration is carried inside t. Let it be clear however that, while the integral curvature exists under very broad conditions, the local curvature may not exist at all as an ordinary function but rather only as a measure.

This last point is essential in defining the curvature of a polyhedron M. It is obvious that if t lies entirely on a face of M we have $\varepsilon_t = 0$. The same result holds if t does not contain any vertex of M. Therefore $K(P) = 0$ if P is not a vertex.

If t contains a vertex, say V, and no other vertices, it will not depend on the explicit form of t but only on the particular choice of V. In other words $\varepsilon_t = \varepsilon_V$ is a characteristic constant of V (the deficency of V) ε_V can be found as follows. Take the sum of all internal angles of the faces of M with vertex V. This sum equals $2\pi - \varepsilon_V$. If several vertices V, X, etc., are inside t we have $\varepsilon_t = \varepsilon_V + \varepsilon_X + \varepsilon_Y + \text{etc.}$

All these results can be condensed into the original formula $\varepsilon_t = \int_t K(P)\,\mathrm{d}A$, provided one understands $K(P)$ as a Dirac type distribution, having the vertices as supports. The integral $\int_M f(P)K(P)\,\mathrm{d}A$, where $f(P)$ is a continuous function, is then to be calculated as $\sum_n f(V_n)\varepsilon_n$, where ε_n is the deficency of V_n, and the summation is carried out on all vertices of M.

If M is a compact (*i.e.* finite closed) polyhedron Gauss-Bonnet's integral curvature theorem can be written as

$$\sum_n \varepsilon_n = 2\pi(2 - N)\,,$$

N is here the genus of M, $N = 0$ for a sphere and $N = 2$ for a torus. There is no loss in generality in supposing that all faces of M are triangles.

Under this form the connection of this theorem with Euler's formula for the genus is most evident. Indeed let σ_{fn} be the internal angle of the face f with the vertex n. We have:

$$\sum_f \sigma_{fn} = 2\pi - \varepsilon_n\,,$$

where the summation is carried out on all faces f having V_n as vertex. It is

also clear that $\sum_n \sigma_{fn} = \pi$, where the summation now is carried out on all vertices belonging to the face f. The sum $\sum_{fn} \sigma_{fn}$ can now be carried out in two different ways by summing first on faces and using then Gauss-Bonnet's theorem or summing on vertices first. With the first method we obtain:

$$\sum_{fn} \sigma_{fn} = 2\pi v - 2\pi(2 - N) .$$

with the second

$$\sum_{fn} \sigma_{fn} = \sum_n \pi - \pi f ,$$

where v, e, f are the number of vertices, edges, faces of M respectively. Since all faces of M are triangles we have $2e = 3f$. Therefore the equation $2v - 2(2 - N) = f$ is simply Euler's formula $v - e + f = 2 - N$.

This example shows how elementary is the treatment of curvature on polyhedra.

Since we are chiefly interested in the intrinsic geometry of manyfolds we are not particularly interested in the edges of M and we regard them as a rather immaterial convention for dividing M into triangles, any other convention being just as good.

It is interesting to notice that the intrinsic geometry of M is completely fixed by the connection matrix and the length of all edges. The connection matrix is essentially a list of all faces, edges and vertices of M and a list of their mutual relationship *i.e.* by reading it one can decide which vertices, edges belong to a given face, etc. The connection matrix supplies us with all the topological information needed in the construction of M.

Once we have constructed a simplectic net on M (in plain words a division of M into triangles) the knowledge of the lengths of all edges of M implies the knowledge of all angles σ_{fn} and therefore of the deficencies ε_n.

The notion of simplectic net replaces therefore the notion of a co-ordinate net. The metric tensor on the other hand is replaced by the lengths of the edges.

If the number of vertices, faces, etc., of M increases the local Gaussian curvature will approximate a continuous function $K(P) = \varrho\varepsilon$, where ϱ is the density of vertices and ε their deficiency, provided this product varies slowly within the test triangle t.

In dealing with higher dimensional analogs of polyhedra a geodesic triangle is no longer convenient in testing curvature. We replace it with the notion of Levi-Civita's parallel transportation (shorthly LCT). Let P and Q be points of M. Let the arc a join P to Q. LCT is then an orthogonal mapping between the vector space S_p in P and the vector space S_q in Q. If in particular $P = Q$

and a is a loop LCT maps S_p into itself. This mapping is a rotation of S around P by some angle $\varepsilon(a)$ which depends on a. By inspection it can be checked that $\varepsilon(a) = \int_a K(P)\,\mathrm{d}A$ and that therefore $\varepsilon(t) = \varepsilon_t$. ε is clearly an additive function of the loop. Take namely two loops a, b with the same end point P. Let us perform LCT around their « product » ab, ab being the loop obtained by sticking end to end the loops a and b. We get $\varepsilon(ab) = \varepsilon(a)+\varepsilon(b)$. On a polyhedron M we shall not consider the case when a vertex lies on a loop because LCT cannot be defined unambiguously.

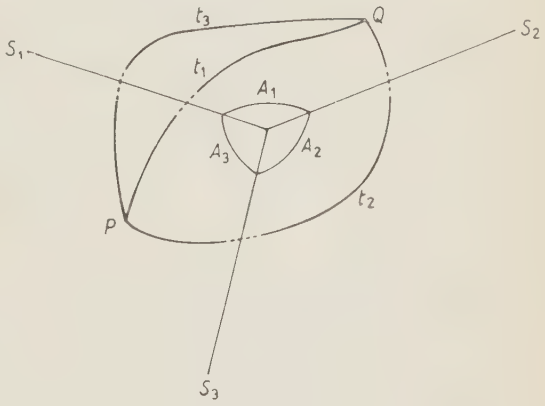

Fig. 1.

If a loop a can be deformed continuously into a loop b by keeping P fixed without encountering the above exceptional situation then $\varepsilon(a) = \varepsilon(b)$, and we write $a = b$. In the following w denotes the set of all vertices of M then $a \approx b$ is simply a homotopy in $M - w$ and is an equivalence relation. A more general and rigorous definition of homotopy will be given later. On the moment it is important to realize that the existence of a curvature set w entails the existence of equivalence classes in the set of loops with a given end point P.

2. – ε-cones.

Let us now consider a polyhedron M with one vertex only. We can visualize M as a one-sheeted cone or pyramid. There is a simple mathematical way of characterizing M. Take namely in the plane polar co-ordinates ϱ and θ so that the metric is given by $\mathrm{d}s^2 = \mathrm{d}\varrho^2 + \varrho^2\mathrm{d}\theta^2$. In the euclidean plane it is understood that two points with the same ϱ and anomalies θ differing by a multiple of 2π are identical. We obtain a manifold with the same intrinsic geometry of a cone by replacing 2π with $2\pi - \varepsilon$.

The origin is then a vertex with deficiency ε. This manifold, hereafter named the ε-cone, is everywhere euclidean with the exception of the vertex.

These ideas can be readily extended to 3 (or higher) dimensional spaces. Let R be the real line of the variable z and let the direct product $R \times \varepsilon$-cone

be metrized as follows:

$$ds^2 = dz^2 + d\varrho^2 + \varrho^2 d\theta^2 .$$

This space is euclidean with the exception of the straight line $\varrho = 0$. We name it the $\varepsilon - 3$-cone. If the product $R^{n-2} \times \varepsilon$-cone is considered with the metric:

$$ds^2 = dz_1^2 + dz_2^2 + \ldots + dz_{n-2}^2 + d\varrho^2 + \varrho^2 d\theta^2 ,$$

we get the $\varepsilon - n$-cone. This manifold is euclidean with the exception of the $n-2$ dimensional flat subset $\varrho = 0$.

The $\varepsilon - n$-cone is the building stone of our structure. The $\varepsilon - 2$-cone is the previously defined ε-cone. If the dimension n is otherwise known we shall drop it from the writing.

3. – Fundamental group.

Take now a path in a Riemannian n-dimensional manifold M_n. We define a path p as a point-valued continuous function $P(s)$ of a parameter s, $0 \leqslant s \leqslant 1$, $P(0)$, $P(1)$ being the end point of the path. If $P(0) = P(1) = P$ then the path is called a loop. We shall regard as identical the loops $P(s)$ and $P(s'(s))$ where $s'(s)$ is non decreasing and continuous and $s(0) = 0$, $s(1) = 1$. If we have two paths a, b, defined by the functions $A(s)$, $B(s)$ such that $A(1) = B(0)$ then the product $c = ab$ is defined by the function $C(s) = A(2s)$ if $s \leqslant 0.5$, $C(s) = B(2s - 1)$ if $s \geqslant 0.5$. The product $c = ab$ joins $A(0)$ to $B(1)$ through $A(1) = B(0)$. This product is associative but in general is not commutative. If a, b are loops with the same end point P then ab and ba both exist and are loops with end point P. In general, however, $ab \neq ba$. a^{-1}, the reverse of a, is defined by $A(1-s)$. Suppose now that our manifold is almost everywhere euclidean except a closed subset w which carries curvature, to be described in detail later. Take two paths a, b with the same end points, defined by $A(s)$ and $B(s)$. We always suppose $B(s) \notin w$, $A(s) \notin w$. If there is a point-valued function $P(s, t)$ of s, $0 \leqslant s \leqslant 1$, and t, $0 \leqslant t \leqslant 1$, continuous in both s and t, such that $P(s, 0) = A(s)$, $P(s, 1) = B(s)$ we say that we have deformed a into b, and the function $P(s, t)$ is called a deformation. If for given a, b it is possible to find a deformation such that $P(s, t) \notin w$ for all s, t then we write $a \sim b$. The symbol $\sim$ satisfies the formal properties of an equivalence (holonomy). Indeed $a \sim a$ because one such a deformation is given by $P(s, t) = A(s)$. If $a \sim b$ then $P(s, 1-t)$ deforms b into a and $P(s, 1-t) \notin w$ it follows $a \sim b$. Similarly if $P(s, t)$ deforms a into b and $Q(s, t)$ deforms b into c and therefore $a \sim b$, $b \sim c$, the function $R(s, t) = P(s, 2t)$ if $t < 0.5$ and $R(s, t) = Q(s, 2t-1)$ if $t > 0.5$ deforms a into c and $R(s, t) \notin w$, $a \sim c$. These definitions apply as well to loops, with the additional requirement that

loops remain such during the deformation or that $P(0, t) = P(1, t)$ for every t. If moreover we have $P(0, t) = P(1, t) = P(0, 0) = P$ then we write $a \approx b$. Clearly $a \approx b$ implies $a \sim b$ but the converse is not true. The symbol $\approx$ applies to loops only.

The symbol $\approx$ also defines an equivalence (homotopy in $M_n - w$). The proof is trivial. If we only know that $a \sim b$ then generally the function $C(s) = P(0, s) = P(1, s)$ defines a loop c because $P(0, 0) = P(0, 1) = P$. Surprisingly enough we have $b\ b \approx c^{-1}ac$. This relation can be checked from the deformation $S(s, t)$ defined as:

$$\begin{aligned} S(s, t) &= C(1 - 3ts) && s < \tfrac{1}{3}, \\ &= P(3s - 1,\ 1 - t) && \tfrac{1}{3} < s < \tfrac{2}{3}, \\ &= C(1 + 3(s - 1)t) && \tfrac{2}{3} < s < 1. \end{aligned}$$

We define next the unit loop u in the point P through the function $U(s) = P = \text{const}$. Clearly $aa^{-1} \approx u$. Suppose now that $a \approx a'$ and $b \approx b'$. Let $P(s, t)$ deform a into a' and $Q(s, t)$ deform b into b'. The function $R(s, t) = P(2s, t)$, $(s < 0.5)$ and $R(s, t) = Q(2s - 1,\ t)$, $(s > 0.5)$ is then a deformation with $R(0, t) = R(1, t) = p$ of ab into $a'b'$ and $R(s, t) \notin w$. Therefore from $a \approx a'$ and $b \approx b'$ it follows $ab \approx a'b'$.

Take now a set of loops with end point P. This set can be partitioned into equivalence classes. A class is identified by one element belonging to it and from now on, unless differently stated, we use a loop to label the whole class. If a and b are classes then all products $a'b'$ where $a \approx a'$ and $b \approx b'$ are homotopic and belong to a new class which we define as the product of the classes a and b and write ab. Moreover if a is a class and u the unit class then $au = a = ua$.

To any class a we can associate the inverse class a^{-1} and $aa^{-1} = a^{-1}a = u$. The product of classes is associative. We see therefore that the set of all classes satisfies the axioms of a group, the fundamental group. Now to each loop a, we can associate an orthogonal matrix $S(a)$ as follows. Take a vector $\boldsymbol{A}$ in P and carry $\boldsymbol{A}$ around a according to LCT. There will be a linear, norm-preserving, mapping between the initial and final position of $\boldsymbol{A}$. This mapping can be therefore represented by an orthogonal matrix if we choose in P a local orthogonal frame of reference. Of course, as we shall require in the applications, if our space is locally pseudoeuclidean we shall have Lorentz matrices. Now the important point is that homotopic loops correspond to the same matrix or, otherwise said, the orthogonal matrix is a function of classes. This in turn implies that the mapping classes matrices are a representation of the fundamental group on the vector space of P. To prove this we first notice that all loops $\approx u$ correspond to unit matrices. If $v \approx u$ then there is a deformation function $T(s, t)$ which carries v into u. The set of all points

$T(s, t)$ is a simply connected surface Σ and we can choose on Σ and in a suitable neighborhood of Σ a globally euclidean frame of reference and the result of L-C-transportation along v calculated in this frame is the identity.

It is furthermore evident that if $c = ab$ and if the matrices $S(a)$, $S(b)$, $S(c)$ correspond to a, b, c we have $S(c) = S(b)\,S(a)$. Take namely a vector $\boldsymbol{V}$ in P and let us carry it along a. The result is by definition $S(a)\boldsymbol{V}$. We then carry this vector along b. We obtain $S(b)\,S(a)\boldsymbol{V}$. But in so doing we have carried $\boldsymbol{V}$ along c, therefore $S(c)\boldsymbol{V} = S(b)\,S(a)\boldsymbol{V}$ and the result follows. If then $a \approx a'$ we know that $a = va'$ where $v \approx u$. It follows $S(a) = S(v)\,S(a')$ but $S(v) = 1$ and $S(a) = S(a')$. The variable in $S(a)$ is then understood to be an equivalence class. The fundamental group has in general infinitely many elements. The example of the ε-cone is instructive. If we take a loop in the ε-cone we can give the angle θ of a point on the loop as function of s. Let $\theta(0) = 0$ without loss of generality. We know that $\theta(1) = N(2\pi - \varepsilon)$. N gives the number of times the loop has encircled the line $\varrho = 0$ or bone, which coincides with the subset w. Loops with the same N are clearly homotopic. Let $a(N)$ be the equivalence class with $\theta(1) = N(2\pi - \varepsilon)$. We have $a(N)\,a(M) = a(N+M)$. The fundamental group is then isomorphic to the group of integers under addition. In order to find $S(a(n))$, shortly $S(n)$ and $S(0) = S$, we notice that $S(N) = S^N$. S will be called the generator of the bone. Take then a vector V in P, and carry it along $a(1)$. V can be split into orthogonal components, one lying into the $\varepsilon - 2$-cone, the other into R^{n-2}. The latter is unaffected by the process, but the first is rotated by the angle ε. The total effect is that of an orthogonal $n \times n$ matrix with R^{n-2} as invariant subspace. In a $\varepsilon - 3$-cone, R^{n-2} is just a real line orthogonal to an $\varepsilon - 2$-cone. S is a rotation with axis $R^{n-2} = R$. This particular fundamental group is abelian but, in general, this is not true.

4. – We shall define now n-dimensional generalizations of polyhedra, shortly skeleton spaces, where curvature is carried by a $n - 2$ subset w, the skeleton. One such structure is the $\varepsilon - n$-cone.

We start from a simplectic decomposition of an n-dimensional space M_n. This decomposition determines the topology of M_n but not the metric. We define a metric in M_n wtih the following axioms.

A) The metric in the interior of any n-dimensional closed simplex T_n is euclidean. This means that we can calculate the distance of any two points inside T_n, define on T_n a cartesian system of co-ordinates, and give the co-ordinates of the points of the boundary of T_n in this frame.

B) In the metric of T_n the boundary of T_n is decomposable into the sum of $n+1$ closed simplexes T_{n-1} and these simplexes are flat.

C) If a simplex T_{n-1} is common boundary of T_n and $T'_{n'}$ the distance of any two points of T_{n-1} is the same in both frames of T_n and $T'_{n'}$.

D) If $P \in T_n$, $P' \in T'_n$ and P, P' are close enough to T_{n-1} we define the distance PP' as the infimum of $PQ+QP'$ for all $Q \in T_{n-1}$.

With this definition the metric is euclidean also on T_{n-1} except perhaps on T_{n-2}'s on the boundary of T_{n-1}. Our definition works on a strictly local scale only but this is all we need to join smoothly neighbouring simplexes. If we want to introduce indefinite metrics a similar but more complicated definition can be used with the same results. Our manifold is now everywhere euclidean outside the union w of all T_{n-2} boundary simplexes (bones). In a M_3 the bones are straight segments while the T_{n-3} are points. Each bone connects two T_0's. If the length of the bones is given, the geometrical structure of the simplexes and of M_3 is determined and in particular their internal dihedral angles can be calculated from elementary formulas of spherical trigonometry. If there would be no curvature on a bone, then the sum of all dihedral angles around the bone should be 2π. But if these lengths are chosen at random, this sum for the i-th bone will be $2\pi - \varepsilon_i$. Here we recognize that the cross section of such a bone is locally an ε_i — 2-cone and that therefore locally the bone is an ε_i — 3-cone. In a S_n the T_{n-2} will carry along locally the same geometry of an $\varepsilon - n$-cone. Take now a T_0 in M_3. In general this point is common end of several bones $T_1^1 \dots T_1^m$. We orient the bones in such a way as to have the positive direction outgoing from T_0. We refer then to T_0 as to an oriented m-joint.

In an m-joint there is a remarkable identity among the $S_p\, p : 1 \dots m$, S_p being the generator of the p-th bone. In order to find it we may deal with an idealized isolated joint, where all departing bones stretch up to infinity. We then select a particular cyclic order of the bones, for instance the one defined by the index p, (0 follows m) the particular choice being irrelevant. We consider then the set of plane sectors, A_p having two contiguous bones T^p, T^{p+1} as sides. We suppose that no two such sectors have common points except perhaps a common boundary bone. These sectors then cut the space in two regions M' and M''. Select the test point P in one of these regions, say M', and a point Q in M''. Take now m paths t_p, joining P to Q, and therefore intersecting the boundary between M' and M''. Let t_p intersect A_p only and only once. Now $t_p t_{p-1}^{-1}$ is a loop a_p with end point P encircling T_1^p once. a_p is a suitable candidate as generator of T_1^p. It must be pointed out however that any other $a'_p \sim a_p$ is just as good and that our procedure merely selects unambiguously one such a_p. We have than the fundamental identity:

$$(1) \qquad \begin{cases} a_1 a_2 a_3 \dots a_p \approx u\,, \\ \text{or} \\ S_1 S_2 \dots S_p = 1\,. \end{cases}$$

The latter identity binds the deficiencies ε_p of the bones and their mutual angles. This can be clearly seen if $m = 3$. In this case:

$$S_1 S_2 S_3 = 1 .$$

This identity can be visualized with the help of a theorem of Euler which states that if in a spherical triangle the sides are in the order the angles of $A_1 A_2 A_3$, then the opposite internal angles are $\pi - \varepsilon_3$, $\pi - \varepsilon_2$, $\pi - \varepsilon_1$. Therefore if we know the deficiency we know the mutual angles of the bones and viceversa.

As we shall see later, in our geometrical approach (1) plays the role of Bianchi's identity in a differentiable Riemannian manifold.

There are corresponding generalizations of (1) in higher dimensional spaces. More precisely there is one such identity at each T_{n-3}, binding the generators of all T_{n-2} bones having this T_{n-3} as boundary. Not all these identities are independent since it is enough to write them for all T_{n-3} having a given T_{n-4} as boundary but one, this one being a consequence of the others. In a sense a T_{n-4} acts as a negative constraint by diminishing the number of Bianchi identities. We have not carried our investigation as far as to show what the actual number of Bianchi's identities is and in particular what is the role of T_{n-5} etc., in determining it since the problem does not arise in M_4.

5. – Differentiable manifolds.

The transition of a skeleton space into a differentiable manifold can be accomplished if we increase the density ϱ of bones while at the same time keeping $\varrho\varepsilon$ slowly varying. We have already shown how this procedure works on a surface. Here we shall proceed in similar fashion on a M_n, $n > 2$. To begin with let's deal with a boundle of parallel bones on a M_3. We suppose that the curvature induced by the bones is small so that we may regards M_3 as approximately euclidean. Let $\boldsymbol{U}$ be a unit vector parallel to the bones. We test the curvature by carrying a vector $\boldsymbol{A}$ around a small loop of area Σ and unit normal $\boldsymbol{n}$, $\boldsymbol{\Sigma} = \Sigma \boldsymbol{n}$. At the end of the test $\boldsymbol{A}$ is found to have rotated around U by the angle $\sigma = N\varepsilon$, where N is the number of bones entangled in the loop. We have $N = \varrho(\boldsymbol{U}, \boldsymbol{\Sigma})$, being the number of bones passing through the loop $\boldsymbol{\Sigma} = \boldsymbol{U}$. The final result is then:

$$\boldsymbol{A} = \varrho\varepsilon(\boldsymbol{U} \wedge \boldsymbol{A})(\boldsymbol{U}, \boldsymbol{\Sigma}) .$$

On the other hand from standard textbooks we know that:

$$A_\mu = R^\sigma_{\varrho,\alpha\beta} \Sigma^{\alpha\beta} A_\sigma ,$$
$$\Sigma^{\alpha\beta} = \varepsilon^{\alpha\beta\gamma} \Sigma_\gamma ,$$
$$\varepsilon^{\alpha\beta\gamma} = \pm 1 = - \varepsilon^{\beta\alpha\gamma} , \quad \text{etc.}$$

By comparing these results we get:

$$R_{\mu\sigma\,\alpha\beta} = \varrho\varepsilon U_{\mu\sigma}U_{\alpha\beta}\,; \qquad U_{\mu\sigma} = \varepsilon_{\mu\sigma\lambda}U^{\lambda}\,. \tag{2}$$

This approximation has all the symmetry properties of the full Riemann tensor. If several bundles are present their effects at this stage are merely additive. In M_n a bone has an orientation determined by a skew symmetric tensor $U_{\mu\sigma}$, $U_{\mu\sigma}U^{\mu\sigma} = 2$, $U_{\mu\sigma}U_{\alpha\beta} + U_{\mu\alpha}U_{\beta\sigma} + U_{\mu\beta}U_{\sigma\alpha} = 0$. If we deal with a bundle of parallel bones we may chose co-ordinates x_1, x_2 perpendicular to the bone, $x_3 \ldots x_n$ parallel to it. In this case $U_{12} = -U_{21} = 1$, while all other components vanish. Eq. (2) still holds provided μ, σ, α, β are allowed to take all values $1 \ldots n$. Independently of n the scalar curvature R is given by $2\varrho\varepsilon$. We shall now show the connection between eq. (1) and Bianchi's identities in this approximation. If the ε's are small (1) can be written as:

$$\sum_1^m \varepsilon_p U^p_{\mu\sigma} = 0 \quad \text{or} \quad \sum_1^m \varepsilon_p \boldsymbol{U}^p = 0\,; \qquad \text{for an } m\text{-joint in } M_3.$$

Suppose next that there is a distribution of identical m-joints of density ϱ. This means also that we have m bundles of parallel bones, the p-th bundle having defect ε_p, direction $\boldsymbol{U}^p$, density ϱ_p. Of course ϱ_p cannot be considered constant because an m-joint produces a decay or creation (depending on the direction) of a bone into or from the others. The rate of decay depends on ϱ. This rate can be estimated as follows. By definition ϱ_p is the density of flux of the outgoing bundle of bones through a surface Σ orthogonal to $\boldsymbol{U}_p$.

Let the position of Σ be determined by an abscissa s computed along the direction of $\boldsymbol{U}_p$. Let C be a cylinder of height $\mathrm{d}s$ and basis $\Sigma(s)$. Each joint in C increases by a unit the flux of ϱ_p bones through $\Sigma(s+\mathrm{d}s)$. There are $\Sigma\cdot\varrho\cdot\mathrm{d}s$ joints in C and therefore:

$$\Sigma\,[\varrho_p(s+\mathrm{d}s) - \varrho_p(s)] = \Sigma\,\varrho(s)\,\mathrm{d}s \qquad \text{or} \qquad \frac{\mathrm{d}\varrho_p}{\mathrm{d}s} = \varrho\,,$$

it follows

$$(\boldsymbol{U}_p\,\mathbf{grad}\,\varrho_p) = \varrho\,.$$

The Riemann tensor is then given by eq. (2) as:

$$R_{\alpha\beta\,\lambda\delta} = \Sigma_p \varepsilon_p \varrho_p U^p_{\alpha\beta}U^p_{\lambda\delta}\,. \tag{3}$$

In our approximation covariant derivatives reduce to normal derivatives. Bianchi's identities can be written as:

$$B_{\alpha\beta\lambda\varepsilon} = \frac{\partial}{\partial x_\varepsilon} R_{\alpha\beta,\gamma\delta} - \frac{\partial}{\partial x_\lambda} R_{\alpha\beta,\gamma\varepsilon} - \frac{\partial}{\partial x_\delta} R_{\alpha\beta,\varepsilon\lambda} = 0\,. \tag{4}$$

We introduce (3) into (4) and treat ε_p and $\boldsymbol{U}_p$ as constants. Use in made of operator identity:

$$U_{\lambda\delta}\frac{\partial}{\partial x_\varepsilon}+U_{\varepsilon\varepsilon}\frac{\partial}{\partial x_\lambda}+U_{\varepsilon\lambda}\frac{\partial}{\partial x_\delta}=(\boldsymbol{U},\,\mathbf{grad})\varepsilon_{\nu\lambda\delta}\,,$$

the result is then:

$$B_{\alpha\beta,\lambda\delta\varepsilon}=\sum_p(\boldsymbol{U}_p,\,\mathbf{grad}\,\varrho_p)\varepsilon_p U^p_{\alpha\beta}\varepsilon_{\lambda\delta\varepsilon}=\varepsilon_{\lambda\delta\varepsilon}\varrho\sum_\mu\varepsilon_\mu U^\mu_{\alpha\beta}=0\,,$$

by virtue of (1). The connection between (1) and Bianchi's identities is then evident.

6. – Lorentz manifolds.

If the metric of a skeleton space is not positive definite then we must replace orthogonal matrices with Lorentz matrices. Correspondingly the defects may become imaginary. Also the usual triangular inequalities among sides of a simplex cease to hold, for instance in a triangle where all sides are timelike one side is larger than the sum of the others. In general, however, it is evident how to modify ordinary trigonometry in order to fit the needs. In an M_4 bones are triangles and, if the metric is indefinite with one time co-ordinate, there are three kinds of bones:

1) Spacelike. Every vector on the bone is spacelike. It is imaginary.

2) Null. Every vector on the bone is a linear combination of a null vector and a spacelike vector orthogonal to it. In this case the corresponding Lorentz matrices cannot be diagonalized. We have $\varepsilon=0$.

3) Timelike. There are both timelike and spacelike vectors on the bone. ε is real.

For simplicity we shall restrict ourselves to timelike bones only or at most null bones.

A useful parameter in the following is the area L of the bone. Let A_μ, B_μ be two sides of the bone. We define L as follows:

$$4L^2=(A_\mu B^\mu)^2-(A_\mu A^\mu)(B_\mu B^\mu)\,.$$

L is therefore always a real quantity. If the bone is timelike, L is real and we choose always $L>0$. We restrict ourselves to timelike bones also in order to avoid ambiguities in the definition of L. If the bone is null then $L=0$.

7. – Variational principles.

The most direct way of approximating an Einstein space with a skeleton space is to use a variational principle. The lagrangian is evidently

$$\mathscr{L} = \frac{1}{16\pi}\int R\,\mathrm{d}^4x\,\sqrt{-g}\,,$$

calculated on the skeleton and we vary this lagrangian in the lengths of all T_1: This choice is particularly happy because we know that these lengths yield the same type of information as the metric tensor. Moreover there is a simple expression for $\mathscr{L}$ in terms of the deficiencies and areas af the bones. Indeed we know that within the T_4 and T_3 there is no contribution to the integral because R is a distribution with the support w. L is an additive function of the bones. Let n label the bones in the skeleton. Then $\mathscr{L} = \sum_n F(T_2^n)$. The function F is of course the same function for all bones. Since the bone is homogeneous, $F(T_2^n)$ is proportional to the area of the bone $F(T_2^n) = L_n f(\varepsilon_n)$, where f depends on ε_n only. T_2^n can be considered as the superposition of two bones $T_2^{n'}$ and $T_2^{n''}$ with the same shape and area and such that $\varepsilon_n = \varepsilon_n' + \varepsilon_n''$. It follows $f(\varepsilon_n) = f(\varepsilon_n') + f(\varepsilon_n'')$. The latter condition implies $f(\varepsilon) = C\varepsilon$, where C is a constant. We may write then $\mathscr{L} = C\sum_n \varepsilon_n L_n$. This result has to be compared with $\mathscr{L} = (1/8\pi)\int \varepsilon\varrho\,\mathrm{d}^4x$ in the limit discussed in Section 5 and for a bundle of identical bones of equal area. From the comparison we get $C = 1/8\pi$ and $\mathscr{L} = (1/8\pi)\sum_n L_n\varepsilon_n$. In M_m the corresponding result is $\mathscr{L} = C\sum_n \varepsilon_n L_n^m$, where L_n^m is the $m-2$ dimensional measure of T_{m-2}^m.

The next task is then the variation of $\mathscr{L}$ in the lengths l_p of T_1^p: While it is clear how L_n depends on l_p, the dependence of ε_n on l_p is rather involved. However quite remarkably we can carry out the variation of $\mathscr{L}$ as if the ε_n were constants:

$$\delta\mathscr{L} = \frac{1}{8\pi}\sum_n \varepsilon_n\,\delta L_n\,,$$

(See the Appendix for the proof; the result holds in any dimension.)

In an M_3 the measures L_p^3 are simply the lengths l_p of T_1^p, which are all independent, the field equations are then simply $\varepsilon_n = 0$, *i.e.* all 3 dimensional Einstein skeletons are flat, as expected.

In M_4 the field equations are instead:

$$\sum_n \varepsilon_n \frac{\partial L_n}{\partial l_p} = 0\,.$$

The summation is now extended on all n such that l_p is a side of L_n. We have. then:

$$\frac{\partial L_n}{\partial l_p} = l_p \operatorname{ctg} \theta_{pn} ,$$

θ_{pn} being the angle which is opposite to l_p in T_2^n. Finally:

$$\sum_n \varepsilon_n \operatorname{ctg} \theta_{pn} = 0 \tag{5}$$

are Einstein's equations for an empty skeleton space. They can be shown, with the methods of Section 5, to degenerate into $R_{\mu\nu} = 0$ for a differential manifold.

If M_4 is a compact manifold, and the number of simplexes in the decomposition is finite, say $1 \leqslant p \leqslant P$, we have P equations (5) in P variables l_p. One such variable can be however fixed at will since if l_p is a solution of (5), also λl_p is a solution. In other words (5) is a set of P homogeneous equations in the l_p's and in general we should have no solution to (5) except the trivial $l_p = 0$. However we have no proof that one or more eq. (5) are not independent of the others and the question of the existence of a compact Einstein skeleton space is still open. Any answer to the question in our simplified model would bring strong evidence in favor of a similar answer for differentiable manifolds. Also (5) is formally suitable for relaxation methods or it could be programmed on a computer if the question of the existence of a solution is first met.

Appendix

We shall show now that for infinitesimal variations of l_i we have in M_m

$$\sum_p \delta\varepsilon_p L_p^m = 0 , \tag{6}$$

L_p^m is the $m-2$ dimensional measure of the n-th bone.

Let T_n be a simplex and let in T_n be defined a cartesian set of co-ordinates. T_n has $n+1$, T_{n-1}^r boundary simplexes, $r : 1 \ldots n+1$. Any two simplexes T_{n-1}^r, T_{n-1}^s have a common boundary T_{n-2}^{rs}. Let U_μ^r be the unit vector normal to T_{n-1}^r, $r : 0 \ldots n-1$, and $U_\mu^r U^{r,\mu} = 1$. If θ_{rs} is the angle between T_{n-1}^r and T_{n-1}^s we have:

$$\cos \theta_{rs} = U_\mu^r U^{s,\mu} .$$

Next we consider the skew symmetric tensor

$$U_{\mu\lambda}^{rs} = - U_{\lambda\mu}^{rs} : \sin \theta_{rs} \, U^{rs} = U_\mu^r U_\lambda^s - U_\lambda^r U_\mu^s .$$

Clearly we have $U^{rs}_{\lambda\mu} U^{rs,\lambda\mu} = 2$ so that $U^{rs}_{\lambda\mu}$ is exactly the tensor used in Section 5, eq. (2) and following, and defines the orientation of T^{rs}_{n-2}. If L_{rs} is the measure of T^{rs}_{n-2} we have the following identity:

$$\sum_s L_{rs} U^{rs} = 0 : \tag{7}$$

take namely a frame where $U^r_0 = 1$, $U^r_{\mu \neq 0} = 0$. Then $V^s_\lambda = U^{rs}_{\lambda 0}$, $\lambda \neq 0$ are the only non vanishing components of $U^{rs}_{\lambda\mu}$ and $V^s_\lambda V^{s,\lambda} = 1$. In T^r_{n-1} the quantity $\sum L_{rs} V^s_\lambda A^\lambda$ is the flux of the constant vector A^λ through the boundary of T^r_{n-1} and therefore vanishes.

Take now the formula:

$$\delta\theta_{rs} = \frac{-1}{\sin\theta_{rs}} (U^r_\mu \delta U^{s,\mu} + U^s_\mu \delta U^{r,\mu}) ,$$

and transform it using the identity:

$$U^{rs}_{\lambda\mu} U^{r,\lambda} = \frac{1}{\sin\theta_{rs}} (\cos\theta_{rs} U^r_\mu - U^s_\mu) ,$$

and:

$$U^{rs}_{\lambda\mu} U^{r,\lambda} \delta U^{r,\mu} = - \frac{1}{\sin\theta_{rs}} (U^s_\mu \delta U^{r,\mu}) ,$$

having used $U^r_\mu \delta U^{r,\mu} = 0$. We get:

$$\delta\theta_{rs} = U^{rs}_{\lambda\mu} (U^{r,\lambda} \delta U^{r,\mu} + U^{s,\lambda} \delta U^{s,\mu}) ,$$

so that by (7):

$$\sum_{r,s} L_{rs} \delta\theta_{rs} = 0 .$$

(6) then follows by taking the sum of the above identity on all simplexes of the skeleton.

* * *

The author wishes to thank Prof. J. A. Wheeler and Dr. C. Misner for many suggestions and encouragement.

RIASSUNTO

In questo lavoro viene sviluppato il formalismo matematico della relatività generalizzata in modo da evitare l'uso di coordinate. A tal uopo si introducono decomposizioni simpliciali di varietà rimanniane che costituiscono l'analogo iperdimensionale di poliedri. Si spera che questo nuovo formalismo renda possibile la discussione di soluzioni delle equazioni di Einstein corrispondenti a topologie di elevata complessità.

IL NUOVO CIMENTO Vol. XIX, N. 3 1° Febbraio 1961

Minimal Electromagnetic Coupling for Spin Two Particles (*).

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(ricevuto il 24 Ottobre 1960)

Summary. — It is noted that if the same Lagrangian that describes the linearized theory of general relativity is extended to charged massed spin 2 particles the coupling cannot be minimal electromagnetic.

There has recently been some revival of interest in field theories of higher spin particles, as developed without the use of subsidiary conditions and indefinite metrics [1-3]. For general relativity this corresponds to the procedure of ref. [2], *i.e.*, the description of gravitational waves in the so-called radiation gauge. In ref. [1] it is shown that for charged spin $\frac{3}{2}$ particles, described by the usual class of possible free Lagrangians and extended to the coupled case by minimal coupling, no such consistent theory is possible. Both Lorentz invariance and positive definiteness are violated. We shall investigate a charged spin 2 theory with minimal coupling and run into still another difficulty.

The Lagrangian we consider is the following:

$$L = \tilde{A}^{\mu\nu}\left[\partial_\alpha \Gamma^\alpha_{\mu\nu} - \tfrac{1}{2}(\partial_\nu \Gamma^\alpha_{\mu\alpha} + \partial_\mu \Gamma^\alpha_{\nu\alpha})\right] - \tilde{\Gamma}^{\beta\nu}_\alpha \Gamma^\alpha_{\nu\beta} + \tilde{\Gamma}^\beta_{\alpha\beta}\Gamma^{\alpha\mu}_\mu - \frac{m^2}{4}\tilde{A}^{\mu\nu}\tilde{A}_{\mu\nu} + \\ + \frac{m^2}{4}\tilde{A}^\alpha{}_\alpha A^\beta{}_\beta + \text{h. c.}$$

$$A^{\mu\nu} = A^{\nu\mu}\,; \qquad \Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}\,.$$

(*) This work is supported in part through AEC Contract AT(30-1)-2098, by funds provided by the U.S. Atomic Energy Commission, the Office of Naval Research and the Air Force Office of Scientific Research.

(1) K. Johnson and E. C. G. Sudarshan: to be published.

(2) R. Arnowitt and S. Deser: *Phys. Rev.*, **113**, 745 (1959).

(3) A. Komar: to be published.

It has the property of leading to a theory of massless particles like gravitons when $m = 0$. (The gauge invariance this implies probably gives a unique specification of this Lagrangian.) When $m \neq 0$ the equations describe spin 2 particles with the correct number of propagating solutions, five for particle and five for antiparticle. However, if the field is coupled to an external electromagnetic field minimally (*i.e.* by the prescription $\partial_n \to \partial_n \pm ieA_n$) in general the number of propagating fields becomes 12 rather than 10. The type of calculations necessary to reach this conclusion are described in ref. (1). By the addition of a term

$$i\frac{e}{2}\tilde{A}^{\mu\nu}F_{\mu}{}^{\beta}A_{\beta\nu},$$

to the Lagrangian the correct number of canonical variables is restored, although other inconsistencies may remain.

In conclusion we note that not only is the minimal electromagnetic coupling not unique, since it depends on the choice of the free Lagrangian, but in at least two theories, the usual spin $\frac{3}{2}$ and spin 2 theories, it leads to inconsistencies. The type of inconsistency induced in spin 2 theory is suggestive of the difficulties that one encounters in the effort to avoid subsidiary conditions; but the lack of a renormalizable theory of these spins keeps the question hypothetical.

RIASSUNTO (*)

Si nota che se lo stesso lagrangiano che descrive la teoria linearizzata della relatività generale viene esteso alle particelle di spin 2 aventi masse con carica, l'accoppiamento non può essere minimo di tipo elettromagnetico.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XIX, N. 3 1° Febbraio 1961

Green's Function in Some Covariant Soluble Problems.

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(ricevuto il 24 Ottobre 1960)

Summary. — Examples of soluble covariant field theory are given, where there is no scattering but where the Green's functions are non-trivial. Exact expressions for the Green's functions are obtained, and in each case they show a peculiar behaviour on the light cone. Some cases exhibit the ghost-like trouble. In one case, it is shown that Lehmann's proof for the spectral representation of the two-body Green's function fails. All examples given here contradict Haag's theorem and the significance of this contradiction is pointed out.

1. – It is well-known that the vector coupling theory of the neutral scalar meson with a fermion field can be solved exactly. Although this theory gives no scattering at all, the Green's functions are not equivalent to those of the free field (1). The purpose of this note is to show that there is an infinite number of soluble theories belonging to a certain class, which are relativistically invariant and give no scattering at all but with non-trivial Green's functions. Exact expressions for these Green's functions have been obtained and they show peculiar singularities on the light cone. Because of this, they in general will not satisfy Lehmann's spectral representation (2) (at least his original proof fails). Another interesting point is the appearance of the ghost-like trouble in one case. This example may give some insight into the ghost problem of the covariant field theory. Finally, all examples considered here violate the so-called Haag theorem and an explanation for this phenomenon is discussed.

(*) This work has been in part supported by National Science Foundation, U. S.

(1) S. Okubo: *Prog. Theor. Phys. (Japan)*, **11**, 80 (1954).

(2) H. Lehmann: *Nuovo Cimento*, **11**, 342 (1954).

2. – We start from the following Lagrangian (*)

$$(1)\qquad \begin{cases} \mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1 , \\ \mathscr{L}_0 = -\dfrac{1}{4}\left[\bar{\psi}\left(\gamma\dfrac{\partial}{\partial x}+m\right)\psi+\psi\left(\gamma^T\dfrac{\partial}{\partial x}-m\right)\bar{\psi}+\text{c.c.}\right]-\dfrac{1}{2}\left[\left(\dfrac{\partial}{\partial x_\mu}\varphi\right)^2+\mu^2\varphi^2\right] \\ \mathscr{L}_1 = \dfrac{i}{4}(\bar{\psi}\gamma_\mu\psi-\psi\gamma_\mu^T\bar{\psi})\dfrac{\partial}{\partial x_\mu}F(\varphi)+\dfrac{i}{4}\dfrac{\partial}{\partial x_\mu}F(\varphi)\cdot(\bar{\psi}\gamma_\mu\psi-\psi\gamma_\mu^T\bar{\psi}) , \end{cases}$$

where $F(\varphi)$ is an arbitrary local hermitian function of $\varphi(x)$. When put we $F(\varphi)\equiv g\varphi$, g being a real constant, then eq. (1) reduces to the vector coupling theory of the neutral scalar meson, which had been treated previously ([1]).

The equations of motion are given by

$$(2)\qquad \begin{cases} \left(\gamma\dfrac{\partial}{\partial x}+m\right)\psi = \dfrac{i}{2}\gamma_\mu\left[\psi\dfrac{\partial F(\varphi)}{\partial x_\mu}+\dfrac{\partial F(\varphi)}{\partial x_\mu}\psi\right], \\ (\Box-\mu^2)\varphi = 0 . \end{cases}$$

The Hamiltonian of our theory is easily computed by the conventional method,

$$(3)\qquad \begin{cases} H = \int \mathrm{d}^3\boldsymbol{x}[H_0(\boldsymbol{x})+H_1(\boldsymbol{x})] \equiv H_0+H_1 \\ H_0(\boldsymbol{x}) = \psi^*(\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta m)\,\psi+\dfrac{1}{2}\left[(\nabla\varphi)^2+\mu^2\varphi^2+\pi^2\right], \\ H_1(\boldsymbol{x}) = -\dfrac{i}{2}\sum_{k=1}^{3}\dfrac{\partial}{\partial x_k}F(\varphi)[\bar{\psi}\gamma_k\psi-\psi\gamma_k^T\bar{\psi}]- \\ \qquad -\dfrac{1}{4}\left(\dfrac{\delta F}{\delta\varphi}\pi+\pi\dfrac{\delta F}{\delta\varphi}\right)(\psi^*\psi-\psi\psi^*)+\dfrac{1}{8}\left(\dfrac{\delta F}{\delta\varphi}\right)^2(\psi^*\psi-\psi\psi^*)^2 , \end{cases}$$

where $\pi(x)$ is the canonical conjugate of $\varphi(x)$, so that

$$\pi(\boldsymbol{x}) = \frac{\partial\varphi}{\partial t}+\frac{1}{2}\frac{\delta F}{\delta\varphi}(\psi^*\psi-\psi\psi^*) .$$

(*) If we use the Hamiltonian equation, as we shall shortly, we can prove that

$$\frac{\partial}{\partial x_\mu}F(\varphi) = \frac{1}{2}\left[\frac{\delta F}{\delta\varphi}\frac{\partial\varphi}{\partial x_\mu}+\frac{\partial\varphi}{\partial x_\mu}\frac{\delta F}{\delta\varphi}\right] .$$

This remark will be usefull in what follows.

For an arbitrary operator Q, we must have the Hamiltonian equation

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = i\,[H, Q]\,. \tag{4}$$

The quantization can be achieved by the canonical commutation relations:

$$\begin{cases} [\pi(\boldsymbol{x}), \varphi(\boldsymbol{x}')] &= -\,i\,\delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')\,, \\ \{\psi(\boldsymbol{x}), \psi^*(\boldsymbol{x}')\}_+ &= \quad \delta^{(3)}(\boldsymbol{x}-\boldsymbol{x}')\,, \\ [\pi(\boldsymbol{x}), \psi(\boldsymbol{x}')] &= \quad [\varphi(\boldsymbol{x}), \psi(\boldsymbol{x}')] = 0\,, \\ [\pi(\boldsymbol{x}), \pi(\boldsymbol{x}')] &= \quad [\varphi(\boldsymbol{x}), \varphi(\boldsymbol{x}')] = 0\,. \end{cases} \tag{5}$$

It is easy to check that our quantization is consistent, *i.e.* the Hamiltonian equation (4) reproduces exactly the Lagrangian eq. (2) on the basis of the commutation laws eq. (5).

Now let us consider the following unitary operator U,

$$U = \exp\left[\frac{i}{2}\int \mathrm{d}^3\boldsymbol{x}\,(\psi^*(\boldsymbol{x})\,\psi(\boldsymbol{x}) - \psi(\boldsymbol{x})\,\psi^*(\boldsymbol{x}))\,F(\varphi(\boldsymbol{x}))\right]. \tag{6}$$

Then, after some algebra, we can easily show that

$$H = UH_0U^{-1} \tag{7}$$

and also we have

$$\begin{cases} U^{-1}\varphi(\boldsymbol{x})\,U = \varphi(\boldsymbol{x})\,, \\ U^{-1}\psi(\boldsymbol{x})\,U = \exp\,[i\,F(\varphi(\boldsymbol{x}))]\cdot\psi\,(\boldsymbol{x})\,. \end{cases} \tag{8}$$

Because of eq. (7), the eigen-functions Ψ_n of H and Φ_n of H_0 are related to each other by the following relation

$$\Psi_n = U\Phi_n\,. \tag{9}$$

Next, we define two time-dependent operators, the Heisenberg operator $Q(t)$ and the free-field operator $Q_0(t)$ (of the interaction representation) respectively, from the time-independent Q as follows:

$$\begin{cases} Q(t) &= \exp\,[iHt]Q\,\exp\,[-iHt]\,, \\ Q_0(t) &= \exp\,[iH_0t]Q\,\exp\,[-iH_0t]\,. \end{cases} \tag{10}$$

Then, using eq. (7) and (8), the Heisenberg operators ψ and φ are related to the free-field operators ψ_0 and φ_0 by

$$(11) \qquad \begin{cases} U^{-1}\varphi(x)U = \varphi_0(x)\,, \\ U^{-1}\psi(x)U = \exp\left[iF(\varphi_0(x))\right]\cdot\psi_0(x)\,, \end{cases}$$

where x denotes a point in the four-dimensional space with the co-ordinates $(\boldsymbol{x}, t)$.

By means of eq. (9) and (11), we can compute the Green's functions exactly. For example, let us consider the two-body case,

$$(12) \qquad G(x, y) = -\left(\Psi_0^* T(\psi(x)\,\bar{\psi}(y))\Psi_0\right).$$

Inserting eq. (9) into eq. (12) and using eq. (11), we obtain

$$(12') \qquad G(x, y) = S_F(x-y)\left(\Phi_0^* P\left(\exp\left[iF(\varphi_0(x))\right],\ \exp\left[-iF(\varphi_0(y))\right]\right)\Phi_0\right),$$

where

$$S_F(x-y) = -\left(\Phi_0^* T(\psi_0(x)\,\bar{\psi}(y))\Phi_0\right)$$

is the free-field propagator.

In order to evaluate the matrix element appearing in eq. (12'), we use the following trick [3]; we express $\exp[iF(z)]$ as a Fourier transform,

$$(13) \qquad \exp\left[iF(z)\right] = \int_{-\infty}^{\infty} \mathrm{d}t\, h(t)\exp\left[itz\right],$$

and use the equation [4]

$$(14) \qquad \left(\Phi_0^* P(\exp\left[it\varphi_0(x)\right],\ \exp\left[-is\varphi_0(y)\right])\Phi_0\right) = \\ = \exp\left[ts\Delta_F(x-y) - \tfrac{1}{2}(t^2+s^2)\Delta_F(0)\right],$$

where Δ_F is the free propagator for the meson field, defined by

$$\Delta_F(x-y) = \left(\Phi_0^* P(\varphi_0(x)\,\varphi_0(y))\Phi_0\right).$$

[3] T. Akiba and K. Sawaba: *Prog. Theor. Phys. (Japan)*, **12**, 94 (1954); C. Goebel: *Phys. Rev.*, **101**, 468 (1956).

[4] R. J. Glauber: *Phys. Rev.*, **84**, 395 (1951).

Thus, finally, we have

$$G(x, y) = S_F(x-y)\int_{\infty}^{\infty} \mathrm{d}t \int_{-\infty}^{\infty} \mathrm{d}s\, h(t)\, h^*(s) \exp\left[ts\, \Delta_F(x-y) - \frac{1}{2}(t^2+s^2)\, \Delta_F(0)\right]. \tag{15}$$

This result can be generalized for all Green's functions. For example, the n-body fermion Green's function is given by

$$\begin{aligned} G(x_1 \ldots x_n; y_1 \ldots y_n) &= G_0(x_1 \ldots x_n; y_1 \ldots y_n) \cdot \\ &\cdot \int_{\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \mathrm{d}t_1 \ldots \mathrm{d}t_n\, \mathrm{d}s_1 \ldots \mathrm{d}s_n\, h(t_1) \ldots h(t_n)\, h^*(s_1) \ldots h^*(s_n) \cdot \\ &\cdot \exp\left[\frac{1}{2}\sum_{i,j=1}^{n}\left[2t_i s_j\, \Delta_F(x_i - y_j) - t_i t_j\, \Delta_F(x_i - x_j) - s_i s_j \Delta_F(y_i - y_j)\right]\right], \end{aligned} \tag{16}$$

where G_0 is the corresponding Green's function of the free field.

For the vector coupling of the scalar meson theory, $F = g\varphi$, g being a real constant. In this case, $h(t) = \delta(g - t)$ and then eq. (16) reduces to the formula which has been derived previously [5] by using a different method. Also, from eq. (16) we can conclude that there is no scattering at all among fermions. We can prove this by using the same method described previously [5]. This conclusion can be easily generalized to include mesons. In this connection, it may be interesting to mention the existence of other theories of a similar nature, *i.e.* the ones having non-trivial Green's functions but which give no scattering at all. One such example has been treated by BARDAKCI and SUDARSHAN [6], whose theory is the one obtained when we keep only finite terms in the so-called Haag expansion of the Heisenberg operator into the infinite series of the free operators. Furthermore, ARAKI *et al.* [7] gave a theorem of a similar nature.

Finally, we would like to point out the meaning of eq. (11). This can be interpreted as follows. In the original Lagrangian eq. (1), we have chosen ψ and q as the fundamental canonical variables. However, if we choose as the canonical variables ψ' and q' such that $q' = q$, $\psi' = \exp[iF(\varphi)]\psi$ from the beginning, then we can see that L reduces to L_0 in terms of the new variables ψ' and q'. This is the situation contained in eq. (11), if we identify ψ' and q' with ψ_0 and q_0, respectively. In the conventional Lagrangian theory, there always exist such ambiguities with regard to the variables which are to

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[5] S. OKUBO: *Nuovo Cimento*, **15**, 949 (1960).

[6] K. BARDAKCI and E. C. G. SUDARSHAN: private communication.

[7] H. ARAKI, R. HAAG and B. SCHROER: preprint *Determination of a local or almost local field from a given current* (1960).

be chosen as the canonical ones. In our case, two different choices of the canonical variables give two different Green's functions, although the physical content is the same, at least, on the surface. However, in some cases a particular choice like the present case will cause some trouble as will be seen in the following section.

3. – In this section, we first investigate two special cases: i) $F=g\varphi$ and ii) $F=g\varphi^2$, corresponding to each case, the Fourier transform of eq. (13) is given by

(i) $$h(t)=\delta(t-g)\,,$$

(ii) $$h(t)=(4\pi|g|)^{-\frac{1}{2}}\exp\left[\frac{\pi}{4}\frac{g}{|g|}\,i-\frac{t^2}{4g}\,i\right].$$

Accordingly the Green's function can be computed from eq. (15),

(17a) (i) $$G(x,y)=S_F(x-y)\exp\left[g^2\Delta_F(x-y)-g^2\Delta_F(0)\right],$$

(17b) (ii) $$G(x,y)=S_F(x-y)[1+4g^2\Delta_F^2(0)-4g^2\Delta_F^2(x-y)]^{-\frac{1}{2}}\,.$$

Therefore, the renormalization constant Z_2 and the renormalized charge g_r^2 are given respectively by

(18a) (i) $$Z_2=\exp\left[-g^2\Delta_F(0)\right],$$

(18b) (ii) $$Z_2=[1+4g^2\Delta_F^2(0)]^{-\frac{1}{2}},$$

(19a) (i) $$g_r^2=g^2\,,$$

(19b) (ii) $$g_r^2=g^2[1+4g^2\Delta_F^2(0)]^{-1}\,.$$

The reason why $g_r^2=g^2$ in the first case can be explained as due to the existence of the Ward identity(5). Finally, the renormalized Green's functions are given by

(20a) (i) $$S_F'(x-y)=S_F(x-y)\exp\left[g_r^2\Delta_F(x-y)\right],$$

(20b) (ii) $$S_F'(x-y)=S_F(x-y)[1-4g_r^2\Delta_F^2(x-y)]^{-\frac{1}{2}},$$

where we have written $G_r(x,y)=S_F'(x-y)$.

We note that in the second case $F=g\varphi^2$, eq. (19b) has exactly the same form as in the Lee model (8) or the Landau model (9), and so generally we have

(8) T. D. Lee: *Phys. Rev.*, **95**, 1329 (1954); G. Källén and W. Pauli: *Kong. Danske Videnskab. Selbskab. Mat.-Fys. Medd.*, **30**, No. 7 (1955).

(9) L. D. Landau, A. Abrikosov and L. Halatnikov: *Suppl. Nuovo Cimento*, **3**, 80 (1956).

a ghost-like situation. This can be best seen from eq. (20*b*), since $S'_F(x-y)\cdot\gamma_4$ would become imaginary for small $(x-y)^2$, thus contradicting the positive definite character of $-G(x,x)\gamma_4$. In view of this difficulty, we are forced to introduce the indefinite metric in our theory. It is quite interesting to observe that eq. (20*b*) is less singular on the light cone than the free field propagator $S_F(x-y)$, as we can see from eq. (20*b*). The same situation does not apply for the first case eq. (20*a*), which is exponentially singular on the light cone. In this case, if we make a formal analytic continuation of g_r^2 into ig_r^2, then eq. (20*a*) exhibits an infinite oscillation near the light cone, which has some similarity to the suggestion given by HEISENBERG ([10]). Such a situation has been already utilized for the renormalizability proof of a certain type of type of theories ([1]) otherwise un-renormalizable in the usual sense.

Now let us investigate the validity of Lehmann's spectral representation ([2]) in this connection. For the second case, eq. (20*b*), it does not hold in the sense that our theory does not admit a positive definite quantization as has been remarked already. For the first case, eq. (20*a*) has an exponential singularity on the light cone, since

$$\Delta_F(x) \sim \frac{1}{4\pi^2}\frac{1}{x^2+i\varepsilon} \qquad (x^2 \sim 0).$$

Thus eq. (20*a*) has a singularity on the light cone, which increases exponentially inside the space-like region. So we may guess that Lehmann's spectral representation cannot hold. Actually, in this case, we will show that indeed, Lehmann's original proof fails. LEHMANN ([2]) has proved that any two-body Green's function $S'^{(\ )}$ satisfies the following representations with the same ϱ_1 and ϱ_2 for all propagators:

$$S'^{(\ )}(x) = S^{(\ )}(x,m) + \int_{(m+\mu)}^{\infty} \mathrm{d}t \left[\varrho_1(t)\left(\gamma\frac{\partial}{\partial x}\right) - \varrho_2(t)\right] \Delta^{(\ )}(x,t)\,, \tag{21}$$

provided the definitions of ϱ_1 and ϱ_2 by means of the following relation have meaning,

$$(i\gamma k)\,\varrho_1(\sqrt{-k^2}) - \varrho_2(\sqrt{-k^2}) = -(2\pi)^3 \sum_n (\Psi_0^* \psi_r(0) \Psi_n)(\Psi_n^* \bar{\psi}_r(0)\Psi_0)\,\delta^{(4)}(k-k_n) =$$

$$= -\frac{1}{2\pi}\int \mathrm{d}^4k \exp\left[-ik(x-y)\right](\Psi_0^* \psi(x)\,\bar{\psi}_r(y)\Psi_0)\,,$$

$$(t\varrho_1(t) \geqslant \varrho_2(t) \geqslant -t\varrho_1(t))\,, \qquad (t>0). \tag{22}$$

([10]) H. P. DÜRR, W. HEISENBERG, H. MITTER, S. SCHLIEDER and K. YAMAZAKI: *Zeits. f. Naturf.*, **14***a*, 441 (1959).

In eq. (22) $\psi_r(x)$ is the renormalized operator of $\psi(x)$. Now the point is the r.h.s. (right-hand side) of eq. (22) does not exist at all in our case and hence ϱ_1 and ϱ_2 are meaningless. Thus, Lehmann's original proof fails at this point. The above mentioned statement can be proved by exactly calculating the integrand on the r.h.s. of eq. (22). By using the method of the previous section, we can easily show that (*)

$$(23) \qquad S'^{(+)}(x-y) = i(\Psi_0^* \psi_r(x)\, \overline{\psi}_r(y) \Psi_0) = S^{(+)}(x, m) \exp\left[ig^2 \Delta^{(+)}(x, \mu)\right],$$

where we have written the S and Δ functions explicitly as functions of m and μ, the nucleon and meson masses respectively. Now eq. (23) has an exponentially divergent singularity on the light cone in the space-like region $x^2 > 0$, since

$$\Delta^{(+)}(x) \sim -\frac{i}{4\pi^2} \frac{1}{\boldsymbol{x}^2 - (x_0 - i\varepsilon)^2}. \qquad (x^2 \to 0):$$

This shows that the r.h.s. of eq. (22) is meaningless, and so the Lehmann's argument fails. Of course, this does not exclude a possibility that the $S'^{(+)}$ function still could satisfy the relations like eq. (21). However, if this were the case, ϱ_1 and ϱ_2 in eq. (21) have to be different for different propagators. This can be shown as follows: Consider the S' function defined by

$$S'(x-y) = i\left(\Psi_0^* (\psi_r(x)\, \overline{\psi}_r(y) + \overline{\psi}_r(y)\, \psi_r(x)) \Psi_0\right).$$

By using the technique of the previous section, we find

$$(24) \quad S'(x) = S^{(+)}(x, m) \exp\left[ig^2 \Delta^{(+)}(x, \mu)\right] + S^{(-)}(x, m) \exp\left[-ig^2 \Delta^{(-)}(x, \mu)\right].$$

The S' function behaves nicely near the light cone, in contrast to the S'_F function. This is due to the fact that the S' function is zero in the space-like region because of the causality. Notice that for the time-like region $x^2 < 0$, $\Delta(x, t)$ can be written as

$$\Delta(x, t) = \frac{t}{4\pi \sqrt{-x^2}}\, \varepsilon(x_0) J_1\left(t\sqrt{-x^2}\right).$$

Then, eq. (21) for the S' function is equivalent to the Fourier-Bessel transform ([11]) of the $S'(x)$ function, so that the weight functions ϱ_1 and ϱ_2 can be

(*) All propagators in this paper are the same as in reference ([2]) except for a factor 2 for the Δ_F and S_F functions.

computed by the inverse Fourier-Bessel formula ([11]). In this way, we get

$$t\varrho_1(t) = \frac{m^2}{2}\int_0^\infty \mathrm{d}s\, sJ_2(st)\left\{H_2^{(2)}(ms)\left[\exp\left|\frac{ig}{8\pi}\left(\frac{\mu}{s}\right)H_1^{(2)}(\mu s)\right] - 1\right] + \right.$$
$$\left. + H_2^{(1)}(ms)\left[\exp\left[\frac{ig^2}{8\pi}\left(\frac{\mu}{s}\right)H_1^{(1)}(\mu s)\right] - 1\right]\right\}.$$

$$\varrho_2(t) = \frac{m^2}{2}\int_0^\infty \mathrm{d}s\, sJ_1(st)\left\{H_1^{(2)}(ms)\left[\exp\left|\frac{ig^2}{8\pi}\frac{\mu}{s}H_1^{(2)}(\mu s)\right] - 1\right] + \right.$$
$$\left. + H_1^{(1)}(ms)\left[\exp\left[\frac{ig^2}{8\pi}\frac{\mu}{s}H_1^{(1)}(\mu s)\right] - 1\right]\right\},$$

From this we find that for $t \to \infty$

$$\varrho_1(t) \sim o(t^{-\frac{3}{2}}), \qquad \varrho_2(t) \sim o(t^{-\frac{1}{2}}).$$

Then we can compute the right hand side of eq. (21) in case of the S_F' function. It is easy to show that then it must have a singularity of at most $(x^2)^{-2}$ near $x^2 \to +0$, whereas eq. (20a) behaves like $\exp[(g^2/4\pi)(x^2)^{-1}]$. This is a contradiction and therefore, if eq. (21) is true, then ϱ_1 and ϱ_2 must be different for different propagators—a situation which is different from the Lehmann's case. Actually, it is quite unlikely that the S_F' function will ever be written in the form of eq. (21). By using a similar argument, we can show that if $\int_0^\infty |\mathrm{d}t|\,\varrho_{1,2}(t)\,|t^{-(n+1)}$ exists for some positive integer n, then the r.h.s. of eq. (21) has to have a singularity of at most $(x^2)^{-n}$ at $x^2 \to +0$, which is again a contradiction with eq. (20a).

Finally, we consider one more singular case,

(iii) $$F(\varphi) = g\log|\varphi| = \tfrac{1}{2}g\log(\varphi^2).$$

In this case, we can still calculate Green's function from eq. (15) to get

(25) $$G(x, y) = \frac{1}{\cosh\left((\pi/2)g\right)}\cosh\left(g\sin^{-1}\frac{\Delta_F(x-y)}{\Delta_F(0)}\right)\cdot S_F(x-y).$$

It is now no longer possible to separate the divergent constants into a multiplicative renormalization factor, but the renormalization constant Z_2 can still be defined by

$$G(x, y) \to Z_2 S_F(x-y), \qquad (x-y)^2 \to \infty.$$

([11]) E. C. TITCHMARSH: *Introduction to the Theory of Fourier Integral*, 2nd Edition (Oxford, 1948), p. 240.

Then we have

$$Z_2 = \frac{1}{\cosh\left((\pi/2)g\right)} .$$

It is remarkable that Z_2 is finite now and satisfies $0 < Z_2 < 1$ as the previous cases.

4. – In this last section we discuss briefly the connection of our theory with the so-called Haag theorem ([12]). Since we have introduced the Heisenberg and interaction representation at $t = 0$ the canonical variables $\psi(x)$, $\varphi(x)$, $\pi(x)$ of the former agree with $\psi_0(x)$, $\varphi_0(x)$, $\pi_0(x)$ of the latter at the time $t = 0$. Or more generally, at an arbitrary time t, we can find an unitary operator V_t such that

$$V_t^{-1}\psi(\boldsymbol{x}, t)V_t = \psi_0(\boldsymbol{x}, t) ,$$

$$V_t^{-1}\varphi(\boldsymbol{x}, t)V_t = \varphi_0(\boldsymbol{x}, t) ,$$

$$V_t^{-1}\pi(\boldsymbol{x}, t)V_t = \pi_0(\boldsymbol{x}, t) ,$$

where V_t is explicitly given by

$$V_t = U \cdot \exp\left[-\frac{i}{2}\int \mathrm{d}^3\boldsymbol{x}\left[\psi_0^*(\boldsymbol{x}, t)\,\psi_0(\boldsymbol{x}, t) - \psi_0(\boldsymbol{x}, t)\,\psi_0^*(\boldsymbol{x}, t)\right]F(\varphi_0(\boldsymbol{x}, t))\right] .$$

Therefore, according to the first part of the Haag theorem ([12]), the two vacuum states Ψ_0 and Φ_0 must agree (notice that at $t = 0$, $V_t = 1$). Therefore, according to eq. (9), we must have $U = 1$, *i.e.* $F = 0$. Then the second part of the Haag theorem ([12,13]) is automatic, *i.e.* our Green's functions must be equivalent to those of the free fields in contrast to eq. (15) and (16). This situation suggests that the operator U is not a proper unitary operator in a given Hilbert space, but a transformation which carries one Hilbert space into another inequivalent one ([14,15]). This can be illustrated also as follows.

([12]) R. Haag: *Kgl. Danske Videnskab. Selbskab. Mat.-Fys. Medd.*, **29**, No. 12 (1955); P. W. Hall and A. S. Wightman: *Kgl. Danske Videnskab. Selbskab. Mat.-Fys. Medd.*, **31**. No. 5 (1957).

([13]) O. W. Greenberg: *Phys. Rev.*, **115**, 706 (1959).

([14]) A. S. Wightman and S. Schweber: *Phys. Rev.*, **98**, 812 (1955); L. Gårding and A. S. Wightman: *Proc. Natl. Acad. Sci. U.S.A.*, **40**, 617, 622 (1954).

([15]) *E.g.* see examples given by R. Haag (reference ([12])) and also by Van Hove: *Physica*, **18**, 145 (1952); J. M. Jauch: *Nuovo Cimento*, **16**, 1068 (1960); Y. Nambu: preprint *Dynamical theory of elementary particles suggested by superconductivity* (University of Chicago).

Define

$$U(\sigma) = \exp\left[\frac{i}{2}\int_{\sigma} \mathrm{d}\sigma_\mu \,(\overline{\psi}\gamma_\mu\psi - \psi\gamma_\mu^T\overline{\psi})\,F(\varphi)\right], \tag{26}$$

where σ is a space-like surface and $U(\sigma)$ is obviously covariant. Then eq. (9) can be re-written as

$$\Psi_n = U(\sigma)\Phi_n\,. \tag{27}$$

Now, since our theory is relativistic invariant, we can choose any reference system for the quantization from the beginning. This means that for a Lorentz transformation including translation, eq. (27) must change as (note that Ψ_n and Φ_n have the same energy and momentum)

$$\Psi_{\tilde{n}} = U(\tilde{\sigma})\Phi_{\tilde{n}}\,, \tag{28}$$

where $\tilde{n}$ and $\tilde{\sigma}$ are the quantities after Lorentz transformation of the state n and the space-like surface σ, respectively. However, the states n cover all possible states and in particular the states $\tilde{n}$. Therefore, eq. (27) must still hold after substituting $n \to \tilde{n}$. Thus we have

$$\Psi_{\tilde{n}} = U(\sigma)\Phi_{\tilde{n}}$$

comparing this with eq. (28), we conclude that $U(\tilde{\sigma}) = U(\sigma)$.

This means that $U(\sigma)$ is independent of the space-like surface σ. Then note that for any two space-like surface σ_1 and σ_2 we have

$$i\int_{\sigma_2}^{\sigma_1} \mathrm{d}^4x(\overline{\psi}\gamma_\mu\psi - \psi\gamma_\mu^T\overline{\psi})\,\frac{\partial F(\varphi)}{\partial x_\mu} = \left(\int_{\sigma_1} - \int_{\sigma_2}\right)\mathrm{d}\sigma_\mu(\overline{\psi}\gamma_\mu\psi - \psi\gamma_\mu^T\overline{\psi})\,F(\varphi) = 0\,,$$

where we have used the fact that $U(\sigma_1) = U(\sigma_2)$. This result is reconcilable only if $F = 0$ (since $F = \mathrm{const}$ is trivial) *i.e.* there is no interaction again. This means that by Lorentz transformation Ψ_n does not go to $\Psi_{\tilde{n}}$ of the same Hilbert space when $F \neq 0$, provided that Φ_n does go to $\Phi_{\tilde{n}}$. Thus in a sense, Ψ_0 is not invariant under a Lorentz transformation, and the Lorentz transformation in general (always except for the pure space rotations and translations) brings one Hilbert space into another inequivalent Hilbert space. This is the reason why we can have Green's functions differing from the free field ones against the Haag theorem.

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The author would like to express his gratitude to Prof. R. E. MARSHAK for discussions and for his kind hospitality at the University of Rochester during the summer of 1960, where a part of this work has been done. He is grateful to Prof. E. C. G. SUDARSHAN for communicating his results to the author.

RIASSUNTO (*)

Si danno esempi di una teoria covariante di campo solubile, in cui non si ha scattering ma nella quale le funzioni di Green non sono banali. Si ottengono espressioni esatte delle funzioni di Green, che in ogni caso presentano un comportamento peculiare nel cono di luce. Alcuni casi danno origine a fenomeni di fantasma. In un caso, si mostra che la prova di Lehmann per la rappresentazione spettrale della funzione di Green per i due corpi fallisce. Tutti gli esempi qui dati contraddicono il teorema di Haag e si sottolinea il significato di questa contraddizione.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XIX, N. 3 1° Febbraio 1961

Generalized Gauge Invariance and Strong Interactions.

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(ricevuto il 28 Ottobre 1960)

Summary. — The principle of generalized gauge invariance is put forward, namely that boson fields and their interactions with the fermion fields be introduced only through demanding the invariance of the free fermion Lagrangian under gauge transformations. If one makes renormalizability a further fundamental criterion to be satisfied by the interactions (obtained through generalized gauge transformations) then, the meson-baryon strong interactions with only pseudo-scalar coupling can be obtained.

1. – Introduction.

The interactions which can be written between fermion and boson fields are fairly large in number. Theoretically, the number is limited by imposing on the interaction Lagrangian certain restrictions, like hermiticity, locality, Lorentz invariance, charge conservation etc. Despite, the imposition of the above requirements and invariance principles, a fair number of boson-fermion interactions are possible. At present there is no way of choosing between them except by appealing to experiment. But, it would be desirable to have some theoretical principle which could limit the choice of interactions offered by the existing well-known principles mentioned above.

In this note we suggest that such a theoretical principle could be provided by the requirement of invariance, under various gauge transformations, of the free fermion Lagrangian L_0. The virtue of this approach is that requirement of invariance of L_0 necessitates the introduction of a boson field as well as its interaction with the fermion field. The nature of the boson field so introduced is then completely determined by appeal to the existing principles like parity conservation, charge conservation etc. Of course if L_0 remains invariant under the transformation then according to our view point the introduction of a

boson field and its interaction with the fermion field would be artificial. The theoretical principle outlined above, referred to as generalized gauge invariance (abbreviated as G.G.I.), has already been suggested [1] by SALAM and GELL-MANN. They further consider renormalizability of the interaction as a further fundamental criterion to obtain uniqueness of the interactions. It is the purpose of this paper to show that the strong and electromagnetic interactions may be generated in a fairly unique way using G.G.I. and renormalizability.

It is well known that the electromagnetic field and its interaction with a charged fermion ψ may be introduced by requiring the free fermion Lagrangian to be invariant under the gauge transformation $\psi \to \exp[ie\alpha(x_\mu)]\psi$. This transformation associates a different rotation α in a two-dimensional charge space with each space-time point. The requirement of invariance demands that physical processes be independent of this arbitrary way of choosing the orientation of charge axes at all space-time points. For a fuller discussion of other gauge transformations one may associate with α see section 2.

Now, if the fermion field ψ is an isotopic spin doublet then one can, in a way analogous to above, associate isotopic gauge transformations [2,3] with a rotation characterized by a vector, $\underline{\alpha}$, in the three-dimensional isotopic space. In I, it was shown that by demanding invariance of L_0 under the infinitesimal isotopic gauge transformation (*)

$$\psi \to (1 + ig\gamma_5 \underline{\tau}\cdot\underline{\alpha})\psi \,, \tag{1}$$

where $\underline{\alpha}$ is a constant with respect to space-time ($\partial_\mu \underline{\alpha} = 0$), it is possible to introduce a spin zero iso-vector boson field $\underline{\varphi}$ and its charge independent pseudoscalar coupling only with ψ. The transformation of $\underline{\varphi}$ simultaneous to (1) is $\underline{\varphi} \to \underline{\varphi} + \underline{\alpha}$ and the gauge is restricted by the condition $\underline{\alpha}\cdot\underline{\varphi} = 0$. Conservation of parity would determine $\underline{\varphi}$ to be pseudoscalar.

In Section 2 we discuss the uniqueness of our procedure for generating the electromagnetic and pion fields and their couplings. In Section 3 we show that the K-meson fields and their couplings to the baryons, in the « doublet approximation » (abbreviated as D.A.) may be generated on the principle of G.G.I.

For this purpose we introduce a new three-dimensional Euclidean space, called H-space, in complete analogy to the usual isotopic spin space. The

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(1) A. SALAM: Lecture, 47th Indian Science Congress, Bombay (1960).

(2) C. N. YANG and R. L. MILLS: *Phys. Rev.*, **96**, 191 (1954).

(3) V. GUPTA: *Nuclear Physics*, **18**, 149 (1960). Hereafter referred to as I.

(*) We follow exactly the notation of I throughout. A bar below denotes isotopic vectors.

relation of the H-space to other spaces introduced by various authors to give a mathematical formulation of the theory of elementary particles, we hope, to discuss in a future publication. But, what we wish to emphasize here is the importance of G.G.I. and renormalizability in generating the boson fields and their strong interactions with the baryons.

2. – Possible gauge transformations and uniqueness of the method of G.G.I.

In this section we discuss the uniqueness of the boson fields and their coupling to fermions generated by the possible gauge transformations in the various spaces corresponding to the intrinsic properties (like charge etc.).

2·1. *Charge space.* – We start with a charged fermion field ψ and its free Lagrangian density L_0 and let α characterize a rotation in the two-dimensional charge space. The possible infinitesimal gauge transformations (giving Lorentz invariance) are

$$\psi \to (1 + ie\alpha)\psi \,, \tag{2a}$$

$$\to (1 + ie\alpha(x_\mu))\psi \,, \tag{2b}$$

$$\to (1 + if\alpha\gamma_5)\psi \,, \tag{2c}$$

$$\to (1 + if\alpha(x_\mu)\gamma_5)\psi \,. \tag{2d}$$

In (2a) and (2c) α is constant with respect to space-time. The transformations (2a) and (2b) are well-known and give charge conservation and the electromagnetic field A respectively. The transformation (2c) permits the introduction of a spin-zero neutral boson field φ, with $\varphi \to \varphi + \alpha$. The invariance of the free boson Lagrangian L_φ requires the boson to be massless otherwise one has to have $\alpha q = 0$! which makes the transformation trivial. Further one has to introduce the non-linear interaction term (in φ)

$$L_I = m\bar{\psi}\psi[\cos(2f\varphi) - 1] - im\bar{\psi}\gamma_5\psi \sin(2f\varphi) \,, \tag{3}$$

making $(L_0 + L_\varphi + L_I)$ invariant under (2c) and $\varphi \to \varphi + \alpha$. Invariance under parity requires φ to be pseudoscalar. The transformation (2d) would permit the field φ with gradient coupling terms in addition to L_I: Renormalizability would rule out (2d) and possibly the transformation (2c) because of the non-linear terms in (3).

2·2. *Isotopic space.* – In the case of a fermion isotopic spin doublet ψ the possible infinitesimal gauge transformations as discussed in I, are

$$(4a) \qquad \psi \to (1 + ig\,\underline{\tau}\cdot\underline{\alpha})\psi\,.$$

$$(4b) \qquad \to (1 + ig\,\underline{\tau}\cdot\underline{\alpha}(x_\mu))\psi\,,$$

$$(4c) \qquad \to (1 + ig\gamma_5\,\underline{\tau}\cdot\underline{\alpha})\psi\,,$$

$$(4d) \qquad \to (1 + ig\gamma_5\,\underline{\tau}\cdot\underline{\alpha}(x_\mu))\psi\,,$$

where in (4*a*) and (4*c*) $\underline{\alpha}$ is constant with respect to space time. The invariance under transformation (4*a*), of L_0, expresses the conservation of isotopic spin while the other three require the introduction of a spin zero boson field $\underline{\varphi}$, with $\underline{\varphi} \to \underline{\varphi} + \underline{\alpha}$, and its coupling terms to ψ. The transformation (4*b*) gives vector coupling, (4*c*) gives only pseudoscalar while (4*d*) gives both pseudoscalar and pseudovector coupling terms. The restrictions placed on $\underline{\alpha}$ for the various couplings are given in I.

The criterion of renormalizability will disallow vector and pseudovector couplings (*) given by the transformations (4*b*) and (4*d*). Thus, our approach yields a pseudoscalar pion field and *only* its pseudoscalar coupling to the fermion doublet. The gauge is restricted by the condition $\underline{\alpha}\cdot\underline{\varphi} = 0$, in isotopic space (**). In the « doublet approximation » (D.A.) one could generate, in general, a different isotopic vector field for each baryon doublet. In particular, we take all four to be identical to the known pion field as indicated by experiment.

2·3. *H-space.* – The gauge transformations in the three-dimensional H-space are exactly analogous to those in isotopic space given by (4) except that ψ is interpreted as a doublet and $\underline{\alpha}$ a rotation in H-space. The components of the H-vector spin zero boson fields generated can be interpreted as the usual K-meson fields (see Section 3).

The nature of the coupling can be again determined to be pseudoscalar by the requirement of renormalizability of the interactions. In the next section we consider only the transformation in H-space which corresponds to (4*c*) so as to satisfy the criterion of renormalizability.

(*) J. Hamilton: *The Theory of Elementary Particles*, p. 292 (1959).

(**) If we attempt to remove the restriction and still demand invariance then one is led to a massless isotopic vector field with non-linear interaction terms as in (3) with φ replaced by $\underline{\varphi}$ and with $\underline{\tau}$, between the ψ's, in the sine term.

3. – Generation of K-meson interactions.

We have seen that the gauge transformations in charge and isotopic spaces essentially yield only the electromagnetic and the pion fields. From the discussion in Section 2, it is clear that to generate interactions between two fermion isotopic doublets we would require gauge transformations which are associated with rotations in some new space.

Consider a three-dimensional Euclidean space H in complete analogy to the usual isotopic spin space. In H-space baryon spinors are constructed with respect to the hypercharge U (= sum of baryon number and strangeness), while in isotopic space the spinors are constructed with respect to the electric charge Q. The transformations in H-space are exactly analogous to those in isotopic space given by (4) except that ψ is now to be interpreted as a doublet and α a rotation in H-space. We will only consider the transformation (in H-space) which corresponds to (4*c*) so as to satisfy the criterion of renormalizability.

Let N_i $(i = 1, 2, 3, 4)$ denote the four baryon isotopic doublets in the notation of PAIS [4]. Now, N_1 has $U = 1$, N_2 and N_3 have $U = 0$ and N_4 has $U = -1$ so that we can construct the four following spinors in H-space:

$$\psi_1 = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}, \quad \psi_2 = \begin{pmatrix} N_3 \\ N_4 \end{pmatrix}, \quad \psi_3 = \begin{pmatrix} N_1 \\ N_3 \end{pmatrix}, \quad \psi_4 = \begin{pmatrix} N_2 \\ N_4 \end{pmatrix}. \tag{5}$$

The free Lagrangian density for the baryons may be written, in terms of ψ_i as $L_0 = L_0^{(1)} + L_0^{(2)}$, where

$$\left\{ \begin{array}{l} L_0^{(1)} = \sum_{i=1}^{4} L_{0i}^{(1)} = \frac{1}{2} \sum_{i=1}^{4} -\frac{1}{2}(\bar{\psi}\gamma_\mu \partial_\mu \psi - \partial_\mu \bar{\psi}\gamma_\mu \psi) \\ \text{and} \\ L_0^{(2)} = \sum_{i=1}^{4} L_{0i}^{(2)} = -\frac{1}{2} m \sum_{i=1}^{4} \bar{\psi}_i \psi_i \,. \end{array} \right. \tag{6}$$

Thus we assume all the baryons to have the same mass « m ». Note that the extra factor one half in $L_0^{(2)}$ and $L_0^{(1)}$, is present so that $L_0^{(2)}$ reduces to $-m \sum_{i=1}^{4} \bar{N}_i N_i$ and similarly for $L_0^{(1)}$. Also, L_0 is invariant under rotations in H-space by construction.

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[4] A. PAIS: *Phys. Rev.*, **110**, 574 (1958).

In complete analogy to isotopic space we require the invariance of L_0 under the infinitesimal gauge transformation

$$\psi_i \to (1 + i\gamma_5 f_i \boldsymbol{\sigma}\cdot\boldsymbol{\beta})\psi_i \tag{7}$$

in H-space, associated with a rotation characterized by $\boldsymbol{\beta}$. The f_i $(i = 1, 2, 3, 4,)$ are real numbers and $\boldsymbol{\sigma}$ are the 2×2 Pauli matrices in H-space. Bold face letters will denote vectors in H-space. We take $\boldsymbol{\beta}$ to be constant in space-time as afore mentioned. Then demanding the invariance of L_0 under (7) it is necessary to introduce a spin zero H-vector field $\boldsymbol{B}^{(i)}(x_\mu) = (B_1^{(i)}, B_2^{(i)}, B_3^{(i)})$ different, in general, for each doublet ψ_i which transforms as

$$\boldsymbol{B}^{(i)} \to \boldsymbol{B}^{(i)} + \boldsymbol{\beta}\,. \tag{8}$$

Further one introduces the interaction terms

$$L_I = im \sum_{i=1}^{4} f_i\, \bar{\psi}_i \gamma_5 \boldsymbol{\sigma}\cdot\boldsymbol{B}^{(i)} \psi_i\,. \tag{9}$$

Then $(L_0 + L_B + L_I)$ is invariant under (7) and (8) under the restriction

$$\boldsymbol{\beta}\cdot\boldsymbol{B}^{(i)} = 0 \tag{10}$$

in H-space. L_B is the sum of the free Lagrangian densities of the fields $\boldsymbol{B}^{(i)}$. The particular case that all the $\boldsymbol{B}^{(i)}$ be identical is not possible if charge is to be conserved (see below). We consider the particular case $\boldsymbol{B}_i^{(1)} = \boldsymbol{B}^{(2)} = \boldsymbol{B}$ and $\boldsymbol{B}^{(3)} = \boldsymbol{B}^{(4)} = \boldsymbol{B}'$ then expanding L_I in terms of N_i, we have (*)

$$\begin{aligned} L_I = im \sum_{i=1}^{4} \bar{N}_i \gamma_5 \varrho_i N_i + \{ & im[f_1 \bar{N}_1 \gamma_5 \sqrt{2}\, B_+ N_2 + f_2 \bar{N}_3 \gamma_5 \sqrt{2}\, B_+ N_4] + \\ & + im[f_3 \bar{N}_1 \gamma_5 \sqrt{2}\, B'_+ N_3 + f_4 \bar{N}_2 \gamma_5 \sqrt{2}\, B'_+ N_4] + \text{h. c.}\} \end{aligned} \tag{11}$$

where

$$\begin{cases} \varrho_1 = (f_1 B_3 + f_3 B'_3)\,, & \varrho_2 = (-f_1 B_3 + f_4 B'_3)\,, \\ \varrho_3 = (f_2 B_3 - f_3 B'_3)\,, & \varrho_4 = -(f_2 B_3 + f_4 B'_3)\,. \end{cases} \tag{12}$$

It is clear that the conservation of the hypercharge (invariance under rota-

(*) We take

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and define

$$B_\pm = (1/\sqrt{2})(B_1 \pm i B_2).$$

tions about the z-axis in H-space) fixes that $U = \pm 1$ for B_+ and $B'_\pm$ while $U = 0$ for B_3 and B'_3. Further, the conservation of charge requires that B_+ be neutral and that B'_+ be positively charged. Moreover, since L_I should conserve isotopic spin the simplest structure, with respect to isotopic space, which the components of $\boldsymbol{B}$ and $\boldsymbol{B}'$ can have is that they be isotopic scalars. Thus if we identify B_+ and B'_+ with the K^0 and K^+-meson fields respectively then L_I, apart from B_3 and B'_3 interactions, yields the usual K-meson-baryon interactions in the doublet approximation of the baryons.

The B_3 and B'_3 represent two neutral isotopic scalar boson fields with zero hypercharge and same mass as the K-mesons. Of course, in particular, we could take them both to be identical. The components B_3 and B'_3 could be taken to be zero if we restrict the β to be perpendicular to the z-axis in H-space. The existence of the fields B_3 and B'_3 can only be decided by experiment. The interesting feature though is that the strength of the B_3 and B'_3 interactions are determined by the four K-meson-baryon coupling constants f_i. Of course the pionic interactions generated via the infinitesimal transformation

$$N_i \to (1 + ig_i \underline{\tau} \cdot \underline{\alpha} \gamma_5) N_i \qquad (i = 1, 2, 3, 4),$$

in isotopic space give another four coupling constants.

To be able to go over from the interactions obtained above in the « doublet approximation » to charge independent interactions in the Gell-Mann–Nishijima scheme one requires $g_2 = g_3$, $f_1 = f_3$ and $f_2 = f_4$. The five independent constants left are still sufficient in number to explain the observed mass-differences between the various baryon isotopic multiplets.

4. – Conclusion.

We have seen that using the principle of generalized gauge invariance it is possible to generate the strong interactions in the D.A. of the baryons from the free baryon Lagrangian.

The pionic interactions are obtained through invariance under gauge transformations in isotopic space and have four independent coupling constants. The K-meson interactions are obtained through the invariance under gauge transformations in H-space and also have another four coupling constants. Further, we require all the baryons to have the same bare mass « m » and the same intrinsic parity. In principle, it is possible to obtain vector, pseudovector and pseudoscalar couplings between the mesons and the baryons. The criterion of renormalizability permits only the pseudoscalar coupling. Then

invariance under the parity transformation requires both the pions and K-mesons to be pseudoscalar with respect to the baryons.

* * *

I am very grateful to Drs. B. Banerjee and S. N. Biswas for helpful comments.

RIASSUNTO (*)

Si propone il principio della invarianza di gauge generalizzata allo scopo di introdurre i campi bosonici e le loro interazioni con i campi fermionici solo richiedendo l'invarianza del Lagrangiano del fermione libero rispetto alle trasformazioni di gauge. Se poi si fa della rinormalizzabilità un ulteriore criterio fondamentale cui le interazioni (ottenute tramite le trasformazioni di gauge generalizzate) devono soddisfare, allora si possono ottenere le interazioni forti mesone-barione con solo accoppiamento pseudoscalare.

(*) *Traduzione a cura della Redazione.*

On the Possible Generalizations of Planck's Law.

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(ricevuto il 16 Luglio 1960)

The present theory of quantized fields, involving only the two universal constants h and c, is incapable of explaining either the mass spectrum of the fundamental particles or the values of the coupling constants. For these and other reasons it is generally agreed that a third dimensional constant l has to be introduced in fundamental theory. There is no agreement, however, how this should be done. Besides non-linear fields, l-quantization of space-time, non-localizable fields, and non-canonical quantization have been proposed, to name but the more promising suggestions. In this situation it is clearly desirable to find a problem that can be treated without *a priori* commitment to any special form of h-c-l-theory.

Such a problem is the spectral distribution of energy in black-body radiation which can be solved, within certain limits, by applying the principles of thermodynamics, adiabatic invariance, and correspondence principles that can be relied upon to survive the transition from the present theory to the future h-c-l-theory. By «correspondence» we mean the requirement that the generalized distribution law degenerates to the Planck law for $l \to 0$.

Apart from the intrinsic interest that attaches to the question as to the generalized form of Planck's law in h-c-l-theory the solution of this problem provides a theoretical criterion to be satisfied by any proposed form of h-c-l-theory. It will also contain some hints as to the direction in which the generalization of the present theory is to be looked for.

1. – We first wish to point out that Wien's displacement law can no longer be relied upon to hold in the generalized theory. Indeed, with the abbreviations

$$x = \frac{h\nu}{kT}, \qquad y = \frac{l}{\lambda} = \frac{\nu}{\nu_0} \tag{1}$$

for the two dimensionless variables we have, from dimensional analysis,

$$\begin{array}{ll} h\text{-}c\text{-theory} & h\text{-}c\text{-}l\text{-theory} \\ \varrho(\nu, T) = A\,\frac{\nu^3 h}{c^3} f(x); & \varrho(\nu, T) = A\,\frac{\nu^3 h}{c^3} F(x, y)\,, \end{array} \tag{2}$$

where A is a number and f and F are numerical functions. The first of these formulae, which contains Wien's displacement law, is clearly equivalent to the statement that y is identically zero, *i.e.*, that $l=0$. As a matter of fact, instead of Wien's displacement law, or the equivalent diff. eq.

$$3\varrho - \nu\varrho_\nu - T\varrho_T = 0\,,$$

one obtains [1]

$$\varrho[1 + G - g - yg_y - TG_T] - g\nu\varrho_\nu - GT\varrho_T = 0\,. \tag{3}$$

Here, G and g are two unknown functions, defined by

$$\frac{\delta\nu}{\nu} = -\,g(y)\,\frac{\delta V}{V} \qquad \text{[for adiabatic expansion]}\,, \tag{4}$$

and

$$\pi(\nu, T) = G(x, y)\,\varrho(\nu, T)\,. \tag{5}$$

$\pi(\nu, T)$ is the spectral density of pressure corresponding to $\varrho(\nu, T)$. Eq. (3) contains all pure thermodynamics can say about our problem. It is only from the field equations that the functions g and G can be determined, apart from

$$g(0) = \tfrac{1}{3} \qquad \text{and} \qquad G(x, 0) \equiv \tfrac{1}{3}\,, \tag{6}$$

as required by correspondence. We shall, however, assume that G is independent of T, *i.e.*

$$G_T = G_x = 0\,. \tag{7}$$

This means that we neglect any possible interaction between quanta belonging to *different* frequencies, so that we still have a semi-ideal gas. Even if (7) should turn out not to hold exactly, we are, for theoretical reasons, primarily interested in the behaviour of quanta belonging to the *same* frequency.

2. – The final result obtained by using the postulates mentioned above reads [1]

$$\varrho(\nu, T) = \frac{8\pi\nu^2}{c^3}\,\chi\,\frac{h\nu\Gamma}{\exp[h\nu\Gamma/kT] - 1}\,, \tag{8}$$

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[1] For proofs see M. STRAUSS: *Monatsberichte der Deutschen Akademie der Wissenschaften zu Berlin*, vol. **2**/7, 412 (1960).

with

$$\chi = \chi[g](y) = (3y^3 g)^{-1} \exp\left[\int_0^y \frac{dy'}{gy'}\right], \tag{9}$$

$$\Gamma = \Gamma[\gamma](y) = y^{-1} \exp\left[\int_0^y \frac{dy'}{\gamma y'}\right], \qquad \gamma = \frac{g}{G}. \tag{10}$$

Thus, Γ depends only on the ratio of the two unknown functions g and G. Hence it is possible that $\Gamma \equiv 1$ without g and G being constant (and hence both being equal to the « classical » value $\frac{1}{3}$); the condition for this to be the case is simply $g(y) = G(y)$. This, then, is the minimum generalization as far as our problem is concerned, but it is not the simplest one from a more general point of view, and hardly the one to be expected (*). The general case where Γ does depend on y will be discussed in a forthcoming paper. Here, we merely wish to point out that Bose statistics is preserved in the generalization (8). This may be taken as an indication that the notions of spin and statistics and their well-known inter-relation are not restricted to the present form of quantum field theory — an inference that is strongly supported by the extension of Pauli's theorem to interacting fields proved by various authors (2).

(*) *Note added in proof.* – M. BORN, in a suggestion for unifying quantum theory and general relativity, obtains a generalization of Planck's law identical with the special case $\chi(y) = 1/\sqrt{1 - y^2}$, $\Gamma(y) \equiv 1$. However, since in this special theory ν_0 (which corresponds to Born's constant b) is clearly the maximum frequency of an oscillator one is not allowed to use the familiar partition function of an oscillator of unrestricted energy [Born's eq. (29)]. If the correct partition function is used, Γ turns out to be a non-analytic function of y. Cfr. M. BORN: *Proc. Roy. Soc.*, A **165**, 291 (1938).

(2) G. LÜDERS and B. ZUMINO: *Phys. Rev.*, **110**, 1450 (1958); also N. BURGOYNE: *Nuovo Cimento*, **8**, 607 (1958).

Interactions of 1.16 GeV/c K^- in Nuclear Emulsions.

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(ricevuto il 17 Settembre 1960)

A stack of 70 sheets of $(20 \times 15 \times \times 0.06)$ cm Ilford G-5 emulsion was exposed to K^--mesons of 1.16 GeV/c momentum. The separated K^- beam from the Berkeley Bevatron was designed and built by Good and Ticho for the Alvarez hydrogen bubble chamber. The stack was exposed behind this chamber during its operation. The beam composition given by the Alvarez bubble chamber groups [1] is:

$$4.5\,\mu^-: 1.5\,K^-: 0.2\,\pi^- .$$

In order to investigate the K^- nuclear interactions, we made a track following scanning with the purpose to have an unbiased sample of interactions, and also an attenuation measurement in order to define the beam composition in the stack and consequently the m.f.p. for negative K.

A total of 225 inelastic interactions were found on 128.5 m of track followed near the entrance part of the beam. Fig. 1 gives the prong distribution of these interactions. Assuming the above beam composition, it turns out that the m.f.p. for nuclear interactions for negative K^- should be ~ 16 cm.

The attenuation measurement was made by counting the interactions at different distances from the entrance of the beam: the stack was divided into four sections and an area scanning method was employed. We counted only stars with primaries making an angle $\leqslant 5°$ with the beam direction. By emplyoing the usual procedures the scanning efficiency was found to be uniform.

The result we obtained for the mixture of interacting π^- and K^- is

$$\lambda = 30 \pm 3 \text{ cm} \quad \text{(see Fig. 2)};$$

the geometrical losses as well as the nuclear and the Coulomb scattering were

(*) A. Caforio and A. Ferilli, Institute of Physics, University of Bari, wish to aknowledge the C.N.R. for a research grant.

(1) L. W. Alvarez, P. Eberhard, M. L. Good, W. Graziano, H. K. Ticho and S. C. Wojcicki: *Phys. Rev. Lett.*, **2**, 215 (1959).

taken into account. The error quoted contains the statistical error only.

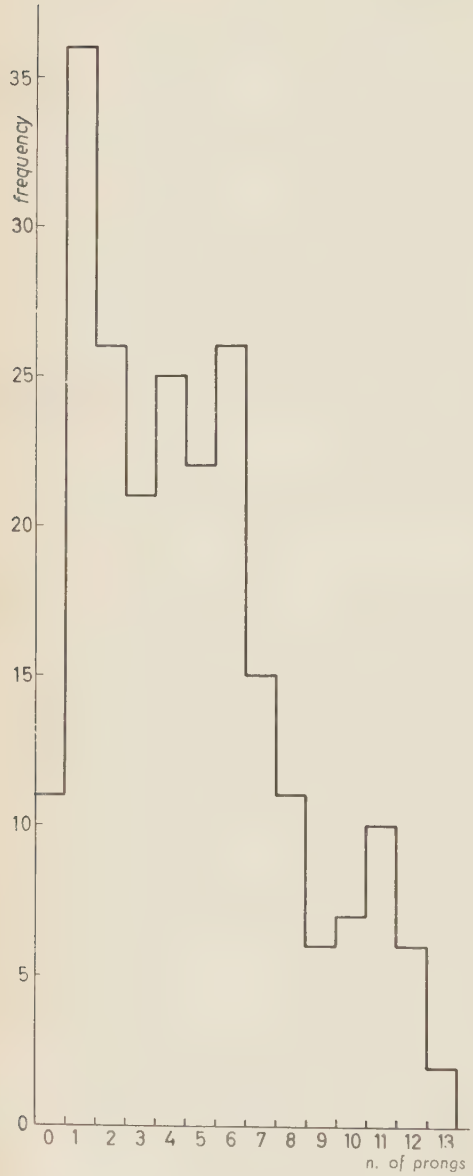

Fig. 1. – Distribution of the prongs of the 225 K^- stars.

A further confirmation of this result was obtained by making a second track-following scanning 14 cm apart from the one already mentioned. In the second region 42 events were detected along 30.7 m of track. On the basis of the above numbers of events the computed m.f.p. for the π^- and K^- mixture evaluated by taking into consideration the correction arising from nuclear scattering, has been found to be $\lambda = 29^{+15}_{-8}$ cm,

It follows that about 50% of the beam is due to K^- and π^--particles. This result apparently supports the conclusion that the contamination due to π^- in our stack is larger than that quoted.

Assuming the m.f.p. for nuclear interactions of the π^- to be 35 cm [2], the ratio of the π^- to the K^- results 3 : 7, and the λ_{K^-} turns out to be about 28 cm, a figure which is in agreement with a total σ K^--nucleon of the order of 50 mb.

Following the preceding considerations we attribute to the K^- the 70% of the 225 inelastic interactions found by the track following scanning.

Table I gives the number of the different types of events in which one or more unstable charged particles are emitted, independently from the associated stable particles.

The figures quoted take into account the correction factors, which are indicated in brackets. These correction factors were evaluated by considering the geometrical losses as well the losses due to K^- and Σ^- captures with zero prongs.

From the above quoted figures it results that:

1) The probability for an interacting K^- to be re-emitted is $\simeq 0.2$.

2) The probability of the emission of a charged Σ is $\simeq 0.15 \pm .04$.

3 The probability for an interacting K^- to produce a π, without ab-

[2] J. C. BRISSON, J. DETOEF, P. FALK-VAIRANT, L. VAN ROSSUM, G. VOLLADAS and LUKE C. L. YUAN: *Phys. Rev. Lett.*, **3**, 561 (1959) H. C. BURROWES, D. O. CALDWELL, D. H. FRISCH: D. A. HILL, D. M. RITSON, R. A. SCHLUTER, and M. A. WAHLIG: *Phys. Rev. Lett.*, **2**, 119 (1959).

TABLE I. – *Summary of the analysis of 225 stars.*

	With unstable particles arising from the stars									With stable prongs only
Type	K^-	$K^+ + \pi$	Σ	$\Sigma + \pi$	HF or Gok	HF $+\pi$ or Gok	Ξ^-	π	$\pi + \pi$	
number	31(4)	7	14(3)	5(1)	1	3	1	46(5)	6	111
	stars certainly due to K^- interactions									

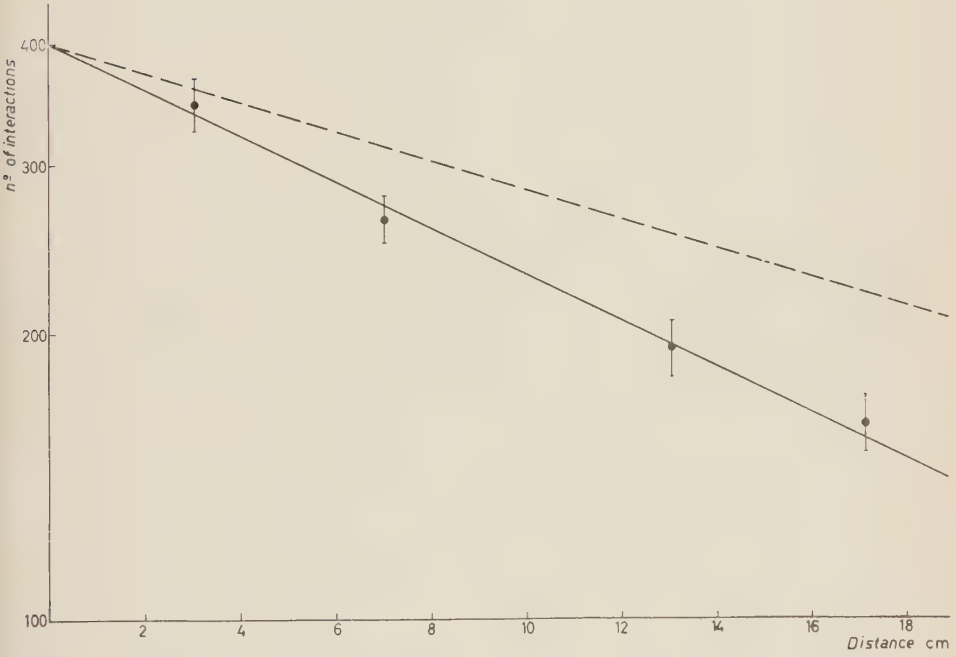

Fig. 2. – Number of interactions at different distances from the entrance of the beam. The full line represents the best fit of the experimental data; the dashed one is the best fit corrected for geometrical losses and for nuclear and Coulomb scattering. The σ_{Coul} was obtained using Rutherford's relativistic formula with a suitable form factor for the charge distribution calculated using the Born approximation. The $\sigma_{nuclear}$ was obtained using the Born approximation with an attractive potential depth of 25 MeV, and $r = r_0 A^{1/3}$ with $r_0 = 1.4 \cdot 10^{-13}$ cm.

sorption of the K^-, is at least 0.20 at our energy.

4) The cross-section for Ξ hyperon production is probably small.

* * *

We would like to thank those who helped to make the experiment possible by exposing the stack, namely Professors M. GOOD, H. TICHO, U. CAMERINI, W. F. FRY and D. J. PROWSE. We are also thankful to Professors N. DALLAPORTA, M. MERLIN and Dr. PATERGNANI for illuminating discussions.

IL NUOVO CIMENTO Vol. XIX, N. 3 1° Febbraio 1961

Dispersion Relation Analysis of P-Wave K-Meson Nucleon Scattering.

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(ricevuto il 2 Dicembre 1960)

1. – Introduction.

Several recent experimental results suggest the existence of possible predominant states in $K\mathcal{N}$ and $\bar{K}\mathcal{N}$ P-wave scattering. The presumably P-state π-meson Λ-hyperon resonance found in production experiments [1], if the K-meson is pseudoscalar (scalar) implies a $T=1$ $P(S)$-wave resonance in the unphysical region of $\bar{K}\mathcal{N}$ scattering. Analysis of K^+p and K^+d scattering in hydrogen and deuterium filled bubble chambers at energies from 50 to 315 MeV [2], confirm previous emulsion results [3] which showed no $T=1$ P-wave scattering and appreciable $T=0$ P-wave scattering.

The success of the now classical Chew, Goldberger, Low and Nambu [4] analysis of low energy pion nucleon scattering suggests an analogous study of the K-meson nucleon case.

2. – Dispersion relations and discussion.

We write the S matrix in the usual way:

$$S = \delta_{fi} - (2\pi)^4 \, i\delta^4(p' + q' - p - q) \sqrt{\frac{N^2}{4EE'\omega\omega'}} \, \bar{u}' T u \,, \tag{1}$$

and the T-matrix in the invariant amplitudes decomposition

$$T = -A(\nu, k^2) + i\gamma \cdot Q \, B(\nu, k^2) \,, \tag{2}$$

(1) M. L. Good: *Tenth Annual Conf. on High Energy Nucl. Phys.* (Rochester, 1960).

(2) S. Goldhaber: *Seminary at Ecole Polytechnique* (Paris, November 1960).

(3) M. A. Melkanoff, D. J. Prowse, D. H. Stork and H. L. Ticho: *Phys. Rev. Lett.*, 108 (1960).

(4) G. F. Chew, M. L. Goldberger, F. E. Low and Y. Nambu: *Phys. Rev.*, **106**, 1337 (1957).

with N being the nucleon mass, p, p', E, E', nucleon initial and final 4-momenta and total c.m. system energy and q, q', ω, ω' the same for the K-meson.

We can write unsubtracted dispersion relations for the invariant amplitudes A and B, if they vanish strongly enough at infinity

$$(3)\qquad \mathrm{Re}\, A^T(\nu, k^2) = P^T_\Lambda \frac{g^2_{r\Lambda}}{2N}\frac{\Lambda - N}{\nu + \nu_\Lambda} + P^T_\Sigma \frac{g^2_{r\Sigma}}{2N}\cdot\frac{\Sigma - N}{\nu + \nu_\Sigma} + \frac{P}{\pi}\int \frac{\mathrm{Im}\, A^T(\nu', k^2)\,\mathrm{d}\nu'}{\nu' - \nu} + \\ + \frac{1}{\pi}\int \sum_{T'} \alpha_{TT'} \frac{\mathrm{Im}\, \bar{A}^{T'}(\nu', k^2)}{\nu' + \nu}\,\mathrm{d}\nu',$$

$$(4)\qquad \mathrm{Re}\, B^T(\nu, k^2) = P^T_\Lambda \frac{g^2_{r\Lambda}}{2N}\cdot\frac{1}{\nu + \nu_\Lambda} + P^T_\Sigma \frac{g^2_{r\Sigma}}{2N}\frac{1}{\nu + \nu_\Sigma} + \frac{P}{\pi}\int \frac{\mathrm{Im}\, B^T(\nu', k^2)}{\nu' - \nu}\,\mathrm{d}\nu' - \\ - \frac{1}{\pi}\int \sum_{T'} \alpha_{TT'} \frac{\mathrm{Im}\, \bar{B}^{T'}(\nu', k^2)}{\nu' + \nu}\,\mathrm{d}\nu',$$

$$(5)\qquad \mathrm{Re}\, \bar{A}^T(\nu, k^2) = -\bar{P}^T_\Lambda \frac{g^2_{r\Lambda}}{2N}\cdot\frac{\Lambda - N}{\nu - \nu_\Lambda} - \bar{P}^T_\Sigma \frac{g^2_{r\Sigma}}{2N}\cdot\frac{\Sigma - N}{\nu - \nu_\Sigma} + \frac{P}{\pi}\int \frac{\mathrm{Im}\, \bar{A}^T(\nu', k^2)}{\nu' - \nu}\,\mathrm{d}\nu' + \\ + \frac{1}{\pi}\int \sum_{T'} \alpha_{TT'} \frac{\mathrm{Im}\, A^{T'}(\nu', k^2)}{\nu' + \nu}\,\mathrm{d}\nu'.$$

$$(6)\qquad \mathrm{Re}\, \bar{B}^T(\nu, k^2) = \bar{P}^T_\Lambda \frac{g^2_{r\Lambda}}{2N}\frac{1}{\nu - \nu_\Lambda} + \bar{P}^T_\Sigma \frac{g^2_{r\Sigma}}{2N}\cdot\frac{1}{\nu - \nu_\Sigma} + \frac{P}{\pi}\int \frac{\mathrm{Im}\, \bar{B}^T(\nu', k^2)}{\nu' - \nu}\,\mathrm{d}\nu' - \\ - \frac{1}{\pi}\int \sum_{T'} \alpha_{TT'} \frac{\mathrm{Im}\, B^{T'}(\nu', k^2)}{\nu' + \nu}\,\mathrm{d}\nu',$$

(the bar meaning anti-K).

With $\nu = -P\cdot Q/N$ and $k^2 = \frac{1}{4}(q-q')^2$, P and Q being $P = \frac{1}{2}(p+p')$, $Q = \frac{1}{2}(q+q')$. We assume the same intrinsic parities for Λ and Σ-hyperons and a negative K-meson hyperon parity and we take the usual isotopic spin assignments $T(\mathrm{K}) = T(\mathcal{N}) = \frac{1}{2}$, $T(\Lambda) = 0$ and $T(\pi) = T(\Sigma) = 1$. Note that, contrary to the pion-nucleon case, the Born approximation does not vanish for the A amplitudes because it is proportional to the mass differences between the initial nucleon and the intermediate one-particle state (hyperon) leading to the pole term. The pole coefficients $P^T_{\Lambda,\Sigma}$, $\bar{P}^T_{\Lambda,\Sigma}$ characteristic of each $K\mathcal{N}$, $\bar{K}\mathcal{N}$ isotopic spin state are:

$$(7)\qquad \begin{array}{c|cc|cc} & T=0 & T=1 & & T=0 & T=1 \\ \hline P^T_\Lambda & +1 & -1 & \bar{P}^T_\Lambda & -2 & 0 \\ P^T_\Sigma & 3 & -1 & \bar{P}^T_\Sigma & 0 & -2 \end{array}$$

and the crossing matrix $\alpha_{TT'}$:

$$(8)\qquad \alpha_{TT'} = \begin{pmatrix} -\frac{1}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \qquad T, T' = 0, 1.$$

From eq. (3) to (6) one obtains in the standard way introduced by CGLN [4] dispersion relations for the partial P-wave amplitudes $h_{T,2J}(\omega)$, defined as

$$h_{T,2J}(\omega) = \frac{\exp[i\delta_{T,2J}]\sin\delta_{T,2J}}{q^3}, \tag{9}$$

ω being the total c.m. energy W minus the nucleon rest mass and q the c.m. momentum of K-meson and nucleon. In the static limit, neglecting the hyperon-nucleon mass differences they reduce to [5]

$$\mathrm{Re}\, h_\alpha(\omega) = \frac{\lambda_\alpha}{\omega} + \frac{P}{\pi}\int \frac{\mathrm{Im}\, h_\alpha(\omega')}{\omega'}\frac{\mathrm{d}\omega'}{\omega} + \frac{1}{\pi}\int \sum_\beta A_{\alpha\beta}\frac{\mathrm{Im}\,\bar{h}_\beta(\omega')}{\omega'+\omega}\mathrm{d}\omega', \tag{10}$$

$$\mathrm{Re}\,\bar{h}_\alpha(\omega) = \frac{\bar{\lambda}_\alpha}{\omega} + \frac{P}{\pi}\int \frac{\mathrm{Im}\,\bar{h}_\alpha(\omega')}{\omega'-\omega}\mathrm{d}\omega' + \frac{1}{\pi}\int \sum_\beta A_{\alpha\beta}\frac{\mathrm{Im}\, h_\beta(\omega')}{\omega'+\omega}\mathrm{d}\omega', \tag{11}$$

α, β running from 1 to 4 as the states $(T, 2J)$ run through 01, 03, 11, 13. The Born approximation coefficients are

$$\lambda_\alpha = \begin{pmatrix} -\left(f_\Sigma^2 + \frac{f_\Lambda^2}{3}\right) \\ +2\left(f_\Sigma^2 - \frac{f_\Lambda^2}{3}\right) \\ -\frac{1}{3}(f_\Sigma^2 + f_\Lambda^2) \\ +\frac{2}{3}(f_\Sigma^2 + f_\Lambda^2) \end{pmatrix}, \qquad \bar{\lambda}_\alpha = \begin{pmatrix} -2f_\Lambda^2 \\ 0 \\ -2f_\Sigma^2 \\ 0 \end{pmatrix} \tag{12}$$

$$f_{\Lambda,\Sigma}^2 = \frac{g_{\Lambda,\Sigma}^2}{(2M)^2},$$

and the crossing matrix

$$A_{\alpha\beta} = \frac{1}{6}\begin{pmatrix} 1 & -4 & -3 & 12 \\ -2 & -1 & 6 & 3 \\ -1 & 4 & -1 & 4 \\ 2 & 1 & 2 & 1 \end{pmatrix}. \tag{13}$$

The one-dimensional dispersion relations from which we started lead to a fairly simple picture of the analytic behaviour of the P-wave amplitudes in the complex ω plane. There are only a pole at the origin corresponding to the Born approximation and two cuts corresponding to the physical regions of the direct and crossed reactions. Double dispersion relations, taking also into account $\mathrm{K}\overline{\mathrm{K}} \to \pi\pi$ reactions lead to a more complicated disposition of the singularities which in the case of

[5] P. K. Roy: *Phys. Rev. Lett.*, **2**, 364 (1959), starts from eq. (10) to (13) but due to errors in the initial equations arrives to no possible $T=0$ $\mathrm{K}N$ resonance.

zero $K\bar{K}\pi\pi$ interaction give our singularities distribution as a good approximation ([6]). The double dispersion relations approach leads among other singularities to a cut from $\omega = -\infty$ to $\omega = m - \{M - \sqrt{M^2 - \mu^2} + m - \sqrt{m^2 - \mu^2}\}$ due to the reaction $K\bar{K} \to \pi\pi$, that is by far the nearest singularity to the physical region.

The contribution of this cut has been assumed by FERRARI, PUSTERLA and FRYE ([7]) to give rise to the « anomalous » low energy behaviour of K^-p elastic scattering. Nevertheless, it has been shown by AMATI, GALZENATI and VITALE ([8]) that this behaviour is predicted by one dimensional forward scattering dispersion relations with no contribution of intermediate $\pi\pi$ states taken into account. Thus, we hope there is no great error in neglecting the $\pi\pi$ cut in the complex ω plane.

We have also neglected, only for simplicity sake, terms in $1/N$ in eq. (10) and (11), this does not affect the qualitative features of our results.

Eq. (10) to (13) are the starting point for the discussion of possible resonant states in scattering problems. From (10) one obtains immediately an effective range type formula:

$$\text{(14)} \qquad \operatorname{Re} h_\lambda(\omega) \quad \frac{\lambda_\lambda}{\omega}\left\{1 + \frac{\omega}{\lambda_\alpha}\left(\frac{P}{\pi}\int \frac{\operatorname{Im} h_\alpha(\omega')}{\omega' - \omega}\,\mathrm{d}\omega' + \frac{1}{\pi}\int \sum_\beta A_{\alpha\beta}\frac{\operatorname{Im}\bar{h}_\beta(\omega')}{\omega' + \omega}\,\mathrm{d}\omega'\right)\right\},$$

or

$$\text{(15)} \qquad \frac{1}{2}\frac{\sin 2\delta_\alpha(\omega)}{q^3} = \frac{\lambda_\alpha}{\omega}(1 + r_\lambda \omega)\,.$$

with the effective range for the phase δ_α

$$\text{(16)} \qquad r_\alpha = \frac{1}{\lambda_\lambda}\left\{\frac{P}{\pi}\int \frac{\operatorname{Im} h_\alpha(\omega')}{\omega' - \omega}\,\mathrm{d}\omega' + \frac{1}{\pi}\int \sum_\beta A_{\alpha\beta}\frac{\operatorname{Im}\bar{h}_\beta(\omega')}{\omega' + \omega}\,\mathrm{d}\omega'\right\},$$

and λ_λ the scattering length.

The most probable candidate for a resonant state will be the one with the largest positive scattering length and effective range.

For the $K\mathcal{N}$ states we shall not consider (11) for the discussion of the possible $\bar{K}\mathcal{N}$ resonance but rather assume the $\bar{K}\mathcal{N}$ resonance induced by the $T=1$, $J=?$ $A\pi$ resonance ([1]) known from production experiments and try to discriminate its unknown J by analysis of its influence on $K\mathcal{N}$ scattering. A glance at eq. (12) permits us to distinguish immediately two different cases depending on the relative strength of $K\Lambda\mathcal{N}$ and $K\Sigma\mathcal{N}$ couplings.

First, let us assume $f_\Sigma^2 < f_\Lambda^2/3$, then from (12) we see that only $K\mathcal{N}$ states (01) and (13) lead to positive scattering lengths. Furthermore, we notice that in this case, $\lambda_{01} < \lambda_{13}$, the Born approximation thus favoring always the $T=1$, $J=\frac{3}{2}$ $K\mathcal{N}$ state. This situation can only be changed if the crossed integral is important. A $T=1$, $J=\frac{1}{2}$ $\bar{K}\mathcal{N}$ resonance would still favor the (13) $K\mathcal{N}$ state, while a $T=1$, $J=\frac{3}{2}$ $\bar{K}\mathcal{N}$ resonance would strongly favor the (01) $K\mathcal{N}$ state due to the large crossing coefficient $A_{(01)(13)} = 2$.

([6]) S. W. MACDOWELL: *Phys. Rev.*, **116**, 774 (1959).
([7]) F. FERRARI, G. FRYE and M. PUSTERLA: *Phys. Rev. Lett.*, **4**, 615 (1960).
([8]) D. AMATI, E. GALZENATI and B. VITALE: *Nuovo Cimento*, **12**, 627 (1959).

In the case, $f_\Sigma^2 > f_\Lambda^2/3$, we also have two positive $K\mathcal{N}$ scattering lengths, that is, in states (03) and as always (13). The relative magnitude of the (03) and (13) scattering lengths depends on the relative magnitude of the $\Lambda K\mathcal{N}$ and $\Sigma K\mathcal{N}$ coupling constants, $f_\Lambda^2 \gtrless f_\Sigma^2$ implying $\lambda_{13} \gtrless \lambda_{03}$. In the case of equal $\Lambda K\mathcal{N}$ and $\Sigma K\mathcal{N}$ couplings, the scattering lengths being equal. Again, if we want a predominant $T=0$ $K\mathcal{N}$ state we have to assume a large crossed integral contribution and from (13) we notice that the crossing matrix always favors a $T=0$ $K\mathcal{N}$ state if a $T=1$ $\bar{K}\mathcal{N}$ state is predominant. The situation is slightly more favorable for a (11) $\overline{K}\mathcal{N}$ resonance than for a (13) $\overline{K}\mathcal{N}$ resonance.

Resuming, our analysis is not inconsistent with a $T=0$ $K\mathcal{N}$ resonance and a $T=1$ $\overline{K}\mathcal{N}$ resonance. Depending on the relative strength of $K\Lambda\mathcal{N}$ and $K\Sigma\mathcal{N}$ couplings we prefer a $J=\frac{1}{2}$ $K\mathcal{N}$ resonance and a $J=\frac{3}{2}$ $\overline{K}\mathcal{N}$ resonance in the case $f_\Sigma^2 < f_\Lambda^2/3$ and a $J=\frac{3}{2}$ $K\mathcal{N}$ resonance and $J=\frac{1}{2}$ (but not excluding $J=\frac{3}{2}$) $\overline{K}\mathcal{N}$ resonance in the $f_\Sigma^2 > f_\Lambda^2/3$ case.

* * *

I am most grateful to Prof. A. Messiah and Drs. M. Jacob, R. Omnès, A. Stanghellini and R. Stora for many useful discussion and advices.

On the Influence of the Formation of Subgrain Structures on the Magnetic Properties of Pure Iron.

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(ricevuto il 3 Dicembre 1960)

Extremely high permeabilities (of the order of 150 000 to 300 000) were already obtained in iron many years ago by Cioffi ([1,2]) by means of a long annealing in pure H_2 at 1480 °C, that is, just below the melting point.

The main effect of this treatment is the elimination of all the non metallic impurities, carbon, nitrogen, oxigen and sulphur, that could produce precipitated particles. Another obvious effect is that of greatly increasing the grain size and eliminating all internal strains.

This high permeability, however, is not always reproducible. In some cases the values obtained are, for some obscure reason, low even if in the specimens it must be excluded that the process of purification in hydrogen is somehow incomplete.

This paper refers to some experiments undertaken to clarify this point. The intent is to examine whether the non exceptional magnetic characteristics observed on some samples could be referred to the presence of subgrain structures inside the crystals. In the experiments referred to herein, it has been shown that these substructures are essentially formed at the $\gamma \rightleftharpoons \alpha$ transition during the cooling of the sample. It appears that they can be eliminated with great difficulty even with prolonged annealing at 880 °C, just below the transformation point. The manner in which the presence of substructures in an otherwise perfect crystal contributes to lowering the permeability of iron even to a large extent, can be readily understood on the basis of the theory of Vicena ([3]) on the influence of dislocations on permeability. In fact this theory demonstrates that the coercive field is proportional to the root of the density of the dislocations present.

A first series of experiments was carried out on samples of Armco iron and electrolytic iron melted in H_2, and a second series on samples of very pure iron reduced in H_2 from iron oxide and melted in H_2 in pure syntherized alumina crucibles. Pure iron oxide was prepared

([1]) P. P. Cioffi: *Phys. Rev.*, **39**, 363 (1932).
([2]) P. P. Cioffi: *Phys. Rev.*, **45**, 742 (1934).
([3]) F. Vicena: *Czechosl. Journ. Phys.*, **5**, 1 (1955).

by extraction of ferric chloride from acid solutions with isopropylic ether [4], precipitation to hydroxide and calcination.

The presence of substructures was determined in each sample on the back reflection Laue diagrams. The X-ray observations were correlated to the results of magnetic permeability measurements. The specimens for magnetic measurements were rings with average diameter of 20 to 30 mm and a section of 15 to 25 mm². The thickness of the rings was always above 3 mm to reduce the influence of surface effects, which contribute to noticeably decreasing the permeability, as Cioffi [1] already pointed out. The Armco iron specimens T 31 A were annealed at 1480 °C for 18 h in dry H_2 (dew point — 60 °C). Nitrogen in the hydrogen was eliminated by passage on Mg at 500 °C. The specimens were then cooled slowly and maintained at 880 °C for 12 h. After such treatment they were generally composed of some 10 large crystals. X-ray measurements were carried out directly on the rings and repeated each time on several crystals. The pure iron specimen from oxide was initially examined after a simple annealing at 880 °C for 18 h. After this treatment maximum permeability in this sample had already reached 201,000. After a subsequent treatment in H_2 at 1470 °C for about 20 h, permeability dropped to 66 500 due to the fragmentation of the crystals into finely divided subgrains slightly misoriented to each other.

After treatment at 1480 °C, the purity of Armco iron samples is markedly improved. A typical analysis of a sample gave: C = 0.004%; S = 0.0043%; N = = 0.0001% plus metallic impurities Mn etc. about 0.03%. Nitrogen and carbon were dosed by internal friction methods, while sulphur content was measured with an analytic method suggested by Luke [5]. The analysis of the sample of iron reduced from oxide gave

C = 0.0014 %; S = 0.001%; N = 0.0001%.

Micrographs revealed complete absence of any type of precipitate.

The first results obtained are collected in Table I: maximum permeability is correlated to the presence of substructures in the crystals as evidenced by the fragmentation of the Laue spots.

Two examples of permeability curves together with the related Laue figures are shown in Figs. 1 and 2. The existence of good qualitative correlation between fragmentation of Laues pots and permeability appears evident. Therefore this fragmentation of the crystals into subgrains seems to be the cause of the poor results that can sometimes be

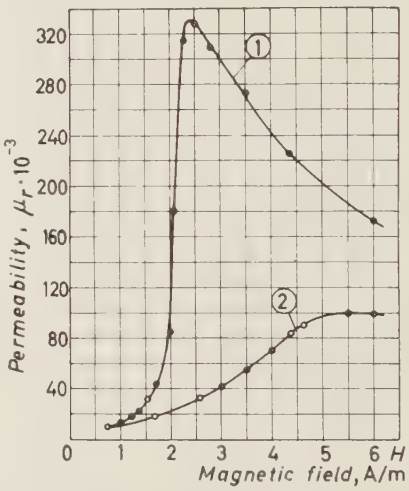

Fig. 1. - Electrolytic iron sp. T 26 A. Permeability *vs.* magnetic field. Curve 1 specimen without substructure corresponding to the Laue diagram in Fig. 2*a*. Curve 2 specimen with substructure corresponding to the Laue diagram in Fig. 2*b*.

[4] R. W. Dodson, G. J. Forney and E. H. Swift: *Journ. Am. Chem. Soc.*, **58**, 2573 (1936).

[5] G. L. Luke: *Ann. Chem.*, **21**, 1369 (1949).

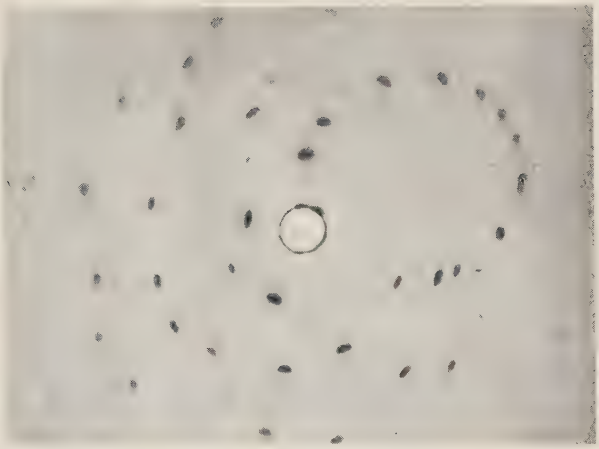

Fig. 2*a*. – Laue pattern of specimen T 26 A annealed 87 h at 1470 °C in hydrogen. Mo target, no filter, distance 3.5 cm ×1.5, substructure absent.

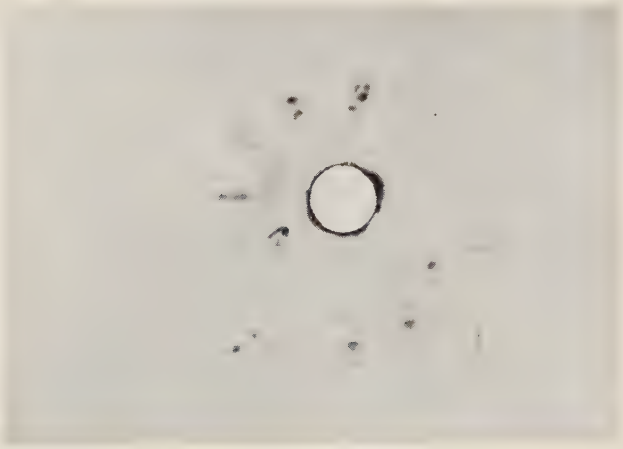

Fig. 2*b*. – Laue pattern of specimen T 26 A annealed (18 + 8) h at 1470 °C in hydrogen. Mo target, no filter, distance 3.5 cm ×1.5, substructure present.

observed in the preparation of iron specimens of very high permeability.

of the following ones. It also appears that once substructures are formed, they

TABLE I.

Specimen	Heat treatment	Fragmentation of the spot on the Laue diagram	Maximum permeability
212 B (Armco)	20 h 1470° + 24 h 880°	very light	118.000
	+ 5′ 950°	severe	60.600
	+ 20 h 1470° + 24 h 880°	severe severe	102.000 102.000
T 31 A (Armco)	18 h 1470° + 15 h 880°	severe	32.400
213 B (Electrolytic)	20 h 1470° + 24 h 880°	no	171.000
	2 passages $\alpha \rightleftharpoons \gamma$ + 66 h 880°	severe	99.000
T-26 A (*) (Electrolytic)	18 h 1470° + 12 h 870°	no	195.000
	+ 8 h 1470° + 12 h 870°	medium	101.000
	+ 87 h 1470° + 12 h 870°	no	330.000
T-36 A (**) (from Fe_2O_3)	18 h 880°	no	201.000
	+ 19 h 1470° + 12 h 850°	severe	74.000

(*) Electrolytic pure iron induction melted in vacuo in sinterized Al_2O_3 crucible.

(**) Pure iron from ferric cloride extracted with isopropilic ether, calcined to Fe_2O_3 and reduced in pure dry H_2 at 600 °C, melted in H_2 in alumina crucible and cooled in the crucible.

An important fact to be stressed is that often the fragmentation seems not to occur during the first cooling of the specimen but much more during one

are extremely stable and cannot usually be eliminated neither with prolonged annealing at 880 °C nor by repeating the treatment at 1480 °C. Substructures

appear to be easier to form and more stable in pure iron reduced from oxide, as the results obtained with sample 36 A prove. These observations seem to be in close relationship with a similar observation made by TALBOT [6] that polygonization in pure iron occurs much more rapidly and is very stable.

Further research work should however be carried out in the future to better clear the mechanism by which substructures depend upon purity of the material used and the way in which transformation occurs.

[6] J. TALBOT: *Colloque International sur les métaux de très haute pureté* (Paris, Oct. 1959).

IL NUOVO CIMENTO Vol. XIX, N. 3 1° Febbraio 1961

Coulomb Field Effects in Bremsstrahlung Processes Associated with β-Decay.

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(ricevuto il 19 Dicembre 1960)

The purpose of this note is to give an abstract of some new results (1) concerning the influence of the nucleus Coulomb field on the internal bremsstrahlung phenomena associated with β-decay.

As is known, large discrepancies between experimental data and the theoretical predictions due to Knipp and Uhlenbeck and to Bloch (2) appeared in the recent experimental results obtained by Starfelt and Svantesson (3) and by Langevin-Joliot (4). One might immediatly think that those discrepancies would be connected with the fact that, in their calculations, Knipp, Uhlenbeck and Bloch neglected the influence of the final nucleus Coulomb field.

Several authors (5,6) have already dealt with the Coulomb effects in the radiative β-decay processes. But in order to keep their calculations in a fairly manageable form, these authors have treated the Coulomb field in an incomplete manner: namely, the Coulomb potential was considered by Lewis and Ford in the Born approximation (see Fig. 1); whereas in the work of Gold

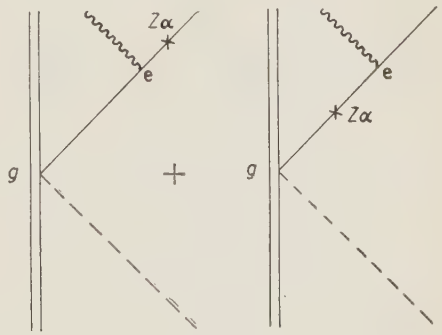

Fig. 1. – Feynman diagrams taken in account by Lewis and Ford (*g* and *e*: respectively the Fermi and electromagnetic coupling constants).

(*) On leave from «Laboratoire de Physique Nucléaire de la Faculté des Sciences», Orsay (S. et O.) where this work had been done.

(1) R. Vinh-Mau: *Thèse* (Paris, 1960). The details of these rather lengthy calculations will be published elsewhere.

(2) J. K. Knipp and G. E. Ulhenbeck: *Physica* **3**, 425 (1936); F. Bloch: *Phys. Rev.*, **50**, 272 (1936) (referred to as KUB).

(3) N. Starfelt and N. L. Svantesson: *Phys. Rev.*, **97**, 708 (1955).

(4) H. Langevin-Joliot: *Ann. de Phys.*, **2**, 16 (1957).

(5) R. R. Lewis and G. W. Ford: *Phys. Rev.*, **107**, 756 (1957).

(6) W. Gold and L. Spruch: *Phys. Rev.*, **113**, 1060 (1959).

and SPRUCH, the β-electron was assumed to be non-relativistic. These authors' results, about the spectrum of γ rays emitted per β-decay, are slightly higher than the KUB ones but deviations from experimental data remain.

At first sight, since the corrections in the 1-st order with respect to $Z\alpha$ appear to be rather small (compared to the deviations between the experimental data and theoretical results) one might expect that the higher order with respect to $Z\alpha$ in the BORN series would be negligible. In fact, for the Coulomb field Born approximation, only the first diagram of Fig. 1 gives rise to a significant contribution, the second one vanishes, so that the Coulomb corrections both for the radiative and non radiative β-decay come only from the modification of the final electron states and a partial cancellation occurs in the ratio of the two processes. Such a situation might not happen in the other terms of the Born series.

In the work of GOLD and SPRUCH, the advantage of the more exact treatment of the Coulomb field is offset by the non relativistic treatment of the β-electrons. However, for the γ-ray energies close to the maximum (*i.e.* the energy range for which the non relativistic approximation is the most accurate) the G.S. values are larger than those of L.F.: this might be interpreted as an indication of a non negligible contribution of the higher order terms.

In the work reported here, we have tried a complete treatment of the nuclear Coulomb effects; so we were led to some mathematical complications due to

1) the relativistic description of the Coulomb effects both in the intermediate and final electron states.

2) the evaluation of the radial integrals.

The first difficulties were overcome by following the GLAUBER and MARTIN [7] method of dealing with Coulomb Green's function and a similar method of DELOFF for wave functions. Those methods allow us to use the standard Casimir technique for determining the transition probabilities.

The difficulties related to the radial integrals arose from the fact that the integrand involved a product of a spherical Bessel's function, a Whittaker's function $W_{a,b}(r)$ and a confluent hypergeometric function, ${}_1F_1(\alpha, \beta; r)$. The origin of each is respectively the photon wave function, the intermediate electron Green's function and the final electron wave function. With the help of a work done by ERDELYI and MEIJER [8] we succeeded in expressing these radial integrals in terms of Horn's 3 variable hypergeometric series. These Horn's generalized hypergeometric series appeared to be very convenient for the numerical applications.

Applications to S^{35}. Results.

We have applied the formulas thus obtained to the case of S^{35}. The numerical computations were performed by an I.B.M. 704 computer. The γ-ray spectrum per β-decay: $R(k)$, the γ-ray circular polarization: $P(k)$, as well as the angular correlations β-γ have been derived.

The results are shown in Tables I and II. We have included in Table I, for comparison, the results from KNIPP and UHLENBECK and BLOCH (K.U.B.), LEWIS and FORD (L.F.), GOLD and SPRUCH (G.S.) and also the experimental results from STAFFELT and SVANTESSON (S.S.) and LANGEVIN-JOLIOT (L.J.).

[7] R. J. GLAUBER and P. C. MARTIN: *Phys. Rev.*, **109**, 1307 (1958).

[8] A. ERDELYI: *Proc. Roy. Soc. Edinburgh* A**62**, 378 (1948).

TABLE I. – $R(k)$: *γ-ray spectrum per β-decay* (*energy units: electron rest mass*).

k	0.05	0.10	0.15	0.20	0.25
K.U.B.	$326 \cdot 10^{-5}$	$73 \cdot 10^{-5}$	$18.1 \cdot 10^{-5}$	$4.1 \cdot 10^{-5}$	$0.56 \cdot 10^{-5}$
L.F.	$324 \cdot 10^{-5}$	$74 \cdot 10^{-5}$	$20.2 \cdot 10^{-5}$	$5.2 \cdot 10^{-5}$	$0.82 \cdot 10^{-5}$
G.S.			$21.4 \cdot 10^{-5}$		$1.06 \cdot 10^{-5}$
V.M.	$324 \cdot 10^{-5}$	$76 \cdot 10^{-5}$	$24.2 \cdot 10^{-5}$	$6.5 \cdot 10^{-5}$	$1.12 \cdot 10^{-5}$
S.S.	$325 \cdot 10^{-5}$	$92 \cdot 10^{-5}$	$30.5 \cdot 10^{-5}$	$8.1 \cdot 10^{-5}$	$1.20 \cdot 10^{-5}$
L.J.	$516 \cdot 10^{-5}$	$146 \cdot 10^{-5}$	$50 \cdot 10^{-5}$	$12.2 \cdot 10^{-5}$	—

TABLE II. – *γ-ray circular polarization* $P(k)$.

k	0.05	0.10	0.15	0.20	0.25
K.U.B.	0.03	0.15	0.30	0.47	0.65
G.S.	—	—	0.23	—	—
V.M.	0.02	0.12	0.26	0.43	0.64

From the results of Tables I, the following significant remarks may be pointed out:

1) the Coulomb effects appear to be important in the radiative β-decay processes: namely at the end of the γ-ray spectrum, the Coulomb corrections are comparable to the K.U.B. values;

2) with respect to the L. F. results ours may be higher by about 30%. Therefore, the partial cancellation of the Coulomb effects in the ratio $R(k)$ which was expected by L.F., seems to be ineffective in the higher order terms of the Born series;

3) the non relativistic approximation for the electrons used by G.S. is good only for $k = 0.25$; for $k = 0.15$, the relativistic effects are approximately 15% and may be more important for $k < 0.15$;

4) by comparison with the experimental data, our results tend to reduce the discrepancy but are still much lower than the results due to LANGEVIN-JOLIOT.

Finally it should be noted that large discrepancies exist between the STARFELT and SWANTESSON data and those from LANGEVIN-JOLIOT.

* * *

The author is deeply grateful to Prof. R. NATAF for suggesting this problem and for his continuous kind guidance.

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Further Remarks on the Proposed μ-e Selection Rule.

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(ricevuto il 7 Gennaio 1961)

In a recent paper published in the *Phys. Rev. Lett.* [1] we proposed a selection rule that forbids transformations of μ into e. We were led to postulating such a selection rule on the basis of the following formal argument.

We assume that μ and e have identical interactions and they differ only in their rest mass. It can then be seen that, if electromagnetic interactions satisfy the requirement of minimality [2] and if weak interactions are neglected, there exists a symmetry property of the theory, that we called μ-e symmetry, related to an exchange of the two lepton fields that are necessary to describe μ and e. We then found that in order to maintain the μ-e symmetry also in a complete theory including weak interactions one has to postulate two neutrinos, one coupled to the electron, ν_e, and one coupled to the muon, ν_μ. This possibility of having two neutrinos has been considered since a long time by different authors on various grounds [3]. Experimentally it seems to be suggested from the absence of $\mu \to e + \gamma$ [4], which should almost certainly be present if in $\mu \to e + \nu + \bar{\nu}$ there is only one kind of neutrino. Now, it is known that to a symmetry property of the theory there corresponds, in general, a physical conservation law. The conservation law corresponding to μ-e symmetry is a

[1] N. Cabibbo and R. Gatto: *Phys. Rev. Lett.*, **5**, 114 (1960); see also R. Gatto: *Proc. of the Annual Intern. Conf. on High Energy Physics* (Rochester), p. 609.

[2] The formal meaning of this requirement is that one can obtain the electromagnetic interactions by substituting $\partial/\partial x_\mu \to \partial/\partial x_\mu - ieA_\mu$ in the original Lagrangian without electromagnetic interactions. The minimality requirement seems to be essential in the present theory of elementary particles. In particular it guarantees the conservation of strangeness in electromagnetic interactions. Theoretically it seems to be related to the condition of renormalizability. In the axiomatic approach to field theory a limitation equivalent to minimality might be directly related to causality — but to our knowledge no systematic investigation of this question has been carried out so far.

[3] A. Salam: *Proc. of the Seventh Annual Conf. on High Energy Nuclear Physics*, 1951 (New York, 1957), p. IX, 44; B. Pontecorvo: *Žurn. Eksp. Teor. Fiz.*, **37**, 1751 (1960); T. D. Lee and C. N. Yang: *Phys. Rev. Lett.*, **4**, 307 (1960).

[4] D. Berley, J. Lee and M. Bardon: *Phys. Rev. Lett.*, **2**, 357 (1959). The experimental situation on the process $\mu^- + (\text{nucleus}) \to \to (\text{nucleus}) + e^-$, whose presence or absence is also directly related to the problem, seems rather confused at present (Conversi *et al.*: to be published; Sard *et al.*: to be published).

multiplicative conservation law that forbids transformations of μ into e.

More specifically, one assigns to each particle a multiplicative muonic quantum number K, according, for instance, to the assignment shown in Table I, and

TABLE I.

particles	quantum number K	quantum number M
μ^-, ν_μ, e^+, $\bar{\nu}_e$	$-i$	-1
μ^+, $\bar{\nu}_\mu$, e^-, ν_e	$+i$	$+1$
mesons, barions, γ	1	0

one obtains that the only reactions that are permitted are those for which the product of the initial K values is equal to the product of the final K values. To the mesons, barions and photon we have assigned $K=1$, but we could have chosen instead $K=\exp[i(\pi/2)N]$ where N is the nucleonic quantum number of the particle, without altering any physical consequence, but only with a slight alteration of the formal argument leading to the postulated law.

Besides the multiplicative conservation law that we have discussed one can consider a more stringent additive conservation law: to each particle there corresponds an additive muonic quantum number M, such that, for instance

$$K = \exp\left[i\frac{\pi}{2}M\right],$$

and only those reactions are allowed for which the sum of the initial M values is equal to the sum of the final M values. Corresponding to our previous assignments of K one can assign values of M as in Table I.

It is obvious that if the additive muonic conservation law is verified also the multiplicative law is verified.

However if the multiplicative muonic conservation law is verified it does not follow that also the additive conservation law is verified. It only follows that the difference ΔM between the sum of the initial M values and the sum of the final M values can only take the values 0, ± 4, ± 8, (*i.e.* $\Delta M = 0$ mod. 4).

Odd ΔM are always forbidden because of lepton conservation. $\Delta M = \pm 2$, ± 6, etc. are forbidden for both types of conservation laws, and this is sufficient to prevent reactions such as

$$\mu + e \to \gamma\,,$$

$$\mu + e \to e + e\,,$$

$$\mu + (\text{nucleus}) \to e + (\text{nucleus})\,,$$

$$\mu \to e + \nu_e + \bar{\nu}_e\,, \quad \pi \to \mu + \nu_e\,, \quad \text{etc.}\,.$$

Reactions with $\Delta M = \pm 4$, ± 8, etc. are forbidden by the additive law, but they are allowed by the multiplicative law. Thus evidence for reactions like, for instance,

(1) $$e^+ + \mu^- \to e^- + \mu^+,$$

(2) $$e^+ + e^+ \to \mu^+ + \mu^+,$$

(3) $$\nu_\mu + (\text{nucleus}) \to \nu_e + e^- + \mu^+ + (\text{nucleus})\,,$$

would exclude the additive conservation law and be consistent with the multiplicative conservation law. None of these reactions can occur by electromagnetic interactions alone and thus any evidence for them would be an evidence for some new mechanism, for instance of the kind we are considering. The reaction (1) is very interesting: it can occur as a charge exchange reaction in muonium

$$(e^+ + \mu^-)_{\text{bound}} \to (e^+ + \mu^+)_{\text{bound}}\,.$$

It has been studied theoretically by PONTECORVO [5], and by FEINBERG and

[5] B. PONTECORVO: *Žurn. Éksp. Teor. Fiz.*, 33, 549 (1957).

WEINBERG [6]. We refer to the work of these authors for further discussion.

The reaction (2) might in principle be studied by colliding beam experiments of the type considered at Stanford [7], but carried out at a much higher energy or with higher intensity. A possible interaction of the type

$$\mathscr{L}' = f(\bar{\psi}^{(\mu)}\gamma'_\lambda a\psi^{(e)})(\bar{\psi}^{(\mu)}\gamma'_\lambda a\psi^{(e)}) ,$$

where

$$a = \tfrac{1}{2}(1+\gamma'_5) \quad \text{and} \quad \bar{a} = \tfrac{1}{2}(1-\gamma'_5) ,$$

through Fierz re-ordering can be written in the form (we use the Majorana representation).

$$\mathscr{L}' = 2f(\psi^{(e)}\gamma'_4\bar{a}\psi^{(e)})(\psi^{(\mu)*}\gamma'_4\bar{a}\psi^{(\mu)*}) .$$

The cross section is then clearly isotropic in the c.m. system. The total cross section is given by

$$\sigma_{\text{total}} = \frac{f^2E^2}{\pi}\beta(1+\beta^2) ,$$

where E is the c.m. energy of each colliding electron and β is the muon velocity in the c.m. system. If, tentatively, we identify f with $\sqrt{8}G$, where G is the weak coupling constant the cross section turns out to be

$$\sigma_{\text{total}} = 1.6\cdot 10^{-37}(E/M)^2 \text{ cm}^2 ,$$

where M is the nucleon mass.

The final muons are polarized longitudinally. The value of the polarization is

$$P = \mp \frac{2\beta}{1+\beta^2} ,$$

where the upper sign holds for μ^+, the lower sign for μ^-.

The reaction (3) can occur through an interaction of the type

$$\mathscr{L}' = f(\bar{\psi}^{(\mu)}\gamma'_\lambda a\psi^{(\nu_e)})(\bar{\psi}^{(\nu_\mu)}\gamma'_\lambda a\psi^{(e)}) ,$$

together with the absorption of a virtual nuclear γ-ray by one of the final charged leptons. An estimate of its cross section on lead gives $0.5\cdot 10^{-39}$ (E_ν in GeV)^2 cm^2 where E_ν is the incident neutrino energy in the laboratory system. The rapid increase of the cross section with energy will perhaps make this process one of the more convenient to decide between the two possibilities of an additive or of a multiplicative muon electron selection rule.

[6] G. FEINBERG and S. WEINBERG: private communication. See the discussion remark by FEINBERG after the paper by R. GATTO in *Proc. of the 1960 Annual Intern. Conf. on High Energy Physics* (Rochester), p. 609.

[7] W. PANOFSKY: *Proc. of the 1960 Annual Intern. Conf. on High Energy Physics* (Rochester), p. 769.

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Recensioni.

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Si tratta di un volume scritto per gli studenti della facoltà di ingegneria. Senz'altro uno dei volumi più indovinati e rispondenti allo scopo che sia capitato di vedere in questa materia anche per gli studenti delle Università italiane.

La distribuzione della materia segue questo ordine. Dopo un capitolo sui concetti fondamentali e sulle definizioni e dopo un capitolo dedicato al I° principio, vengono tre capitoli dedicati alle proprietà delle sostanze, diagrammi di stato, gas ideali, gas reali. Successivamente sono introdotti il secondo principio, i concetti di reversibilità e irreversibilità, temperatura termodinamica, entropia. Seguono altri capitoli sulle proprietà fisiche, energia utilizzabile, miscele di gas e vapori, la combustione, equilibrio chimico in reazioni di gas ideali, aspetti termodinamici del moto dei fluidi e macchine a fluido.

Nell'ultima parte sono trattati argomenti più applicativi: cicli a gas, cicli a vapore, refrigerazione. Gli ultimi due capitoli riguardano cenni su miscele binarie e sulla trasmissione del calore.

La disposizione degli argomenti risponde a un criterio più pedagogico che logico. L'intento di fare un'opera didattica è comunque chiaro in ogni pagina, Si noti l'insistenza con cui è distinto il sistema chiuso dal sistema aperto con deflusso continuo, la chiarezza con cui si sottolinea la differenza fra $\int p\,dv$ e $-\int v\,dp$ e il lavoro utile esterno netto nei processi irreversibili, la precisione dei vari esempi numerici completamente svolti che accompagnano lo sviluppo di ogni questione.

Alla fine di ogni capitolo un cenno riassuntivo, una essenziale, ma precisa, indicazione bibliografica e un gran numero di problemi proposti (in tutto 700).

Molti argomenti sono tralasciati (es.: effetto termoelettrico, relazioni di Onsager, ecc.), altri, quelli dell'ultima parte del volume e in particolare la trasmissione del calore sono trattati in modo ridotto e riassuntivo. Non si prescinde dalla necessità di trattazione ulteriore degli argomenti specifici nell'educazione dell'allievo ingegnere. Il volume porta infatti come sottotitolo « An introductory text book ».

G. Mattarolo

H. A. Strobel – *Chemical Instrumentation*, 653 pp., Addison-Wesley Publishing Company Inc., Reading, Massachusetts (USA), 1960, $ 9,75.

Nell'analisi chimica, soprattutto se quantitativa, si vanno sempre più diffondendo ed affermando tutta una serie di metodi fisici di misura e controllo, facenti uso di un'apparecchiatura varia e talora complessa. Questa che l'autore chiama la linea di sviluppo strumentale dell'analisi chimica e che si distacca talvolta notevolmente dalla linea tradizionale, richiede a chi ne vuol utilizzare i sistemi una conoscenza precisa delle tecniche usate e dei principi su cui esse si basano, nonchè una discreta capacità di progettare l'apparecchiatura più adatta allo scopo prefissato, usando strumenti le cui caratteristiche di disegno e funzionamento siano ben comprese e sfruttate.

In questo volume della serie di chimica della Casa Editrice Addison-Wesley, l'autore, dott. H. A. Strobel, si propone di fornire a quanti sono ingaggiati nella ricerca nel campo dell'analisi chimica quantitativa, appunto una tale

conoscenza delle tecniche e dei principi dell'analisi strumentale.

Nel primo capitolo vengono esaminati i criteri ed i concetti generali di progettazione ed uso di sistemi strumentali a più componenti.

Il secondo capitolo è dedicato allo studio degli errori di misura, alla loro valutazione mediante le misure di precisione, allo studio della propagazione degli errori nelle misure indirette. Con ciò si chiude la parte introduttiva generale dell'opera.

La seconda parte si occupa dei metodi di analisi facenti uso di misure ottiche.

I capitoli terzo e quarto vengono impiegati per esporre i fondamenti fisici dell'ottica, e sono dedicati rispettivamente, il capitolo terzo, alla natura delle radiazioni elettromagnetiche ed alle loro interazioni con la materia, il capitolo quarto, ai fenomeni di interferenza e polarizzazione, nonchè all'ottica geometrica.

Il quinto capitolo si occupa della spettroscopia di emissione come tecnica analitica. Vi si studiano in dettaglio i metodi di eccitazione dei materiali da analizzare, gli spettrografi a prisma ed a reticolo, gli inerenti problemi di rivelazione e registrazione, e le tecniche di preparazione dei campioni. Un esame degli strumenti, delle tecniche preparative, ecc., altrettanto completo viene riservato nel capitolo sesto alla fotometria di assorbimento ed alla spettrofotometria.

Nel capitolo ottavo vengono prese in esame le tecniche fondate su misure di diffusione della luce e sull'impiego dell'effetto Raman.

Il capitolo nono è dedicato all'uso della misura dell'indice di rifrazione di un preparato come strumento analitico.

La rotazione ottica ed i relativi apparecchi di misura come mezzi di riconoscimento e determinazione quantitativa di sostanze, vengono studiati nel capitolo decimo. Questo capitolo conclude la seconda parte del libro.

La terza parte è interamente dedicata all'esposizione dei metodi di analisi elettrici. Anche ora come all'inizio della seconda parte, i primi capitoli servono ad esporre le fondamentali nozioni di fisica e di elettronica, necessarie a ben capire i metodi e le apparecchiature descritte nel seguito.

Più precisamente, il capitolo undicesimo tratta dei fenomeni elettrici in generale e dei processi connessi con la migrazione ionica in liquidi e gas.

Il capitolo dodicesimo è dedicato all'esposizione dei primi fondamenti di elettronica, sia con circuiti impieganti diodi, triodi e pentodi termoionici, sia con i più recenti circuiti transistorizzati.

Nel tredicesimo capitolo infine si affronta l'elettronica un po' più avanzata degli amplificatori a corrente continua, degli oscillografi a raggi catodici, e dei circuiti integratori e differenziali.

Nel capitolo quattordicesimo vengono esaminati i metodi analitici basati sulle misure di conduttività elettrica e di mobilità ionica in liquidi e gas. Vengono anche esaminati metodi di analisi dei gas fondati sulla misura della conduttività termica dei gas (il pretesto per parlarne nella parte dedicata alle misure elettriche sta nell'uso dei metodi elettrici impiegati per la determinazione della conduttività termica).

Il capitolo quindicesimo si interessa dei metodi di analisi quantitativa fondati sulla misura della forza elettromotrice di opportune celle voltaiche. Importanza particolare fra questi metodi potenziometrici assumono i pH-metri, di cui è fatta una esauriente trattazione.

Il capitolo sedicesimo è dedicato ai metodi polarografici, ed il diciassettesimo ai metodi elettrogravimetrici e coulometrici. Nei primi la variabile che viene osservata è la corrente ionica (che nei metodi potenziometrici era nulla o quasi), mentre nei secondi si determina la quantità di elettricità passata attraverso la cella elettrolitica. In ciascun caso vien posto l'accento sulle apparecchiature usate e sulla loro progettazione.

Infine il capitolo diciottesimo è riservato alla trattazione dei sensibilissimi recenti metodi di analisi facenti uso di radioisotopi e delle svariate applicazioni dei radioisotopi come traccianti in chimica.

Chiude il libro una serie di 24 complete descrizioni di misure che il lettore dovrebbe eseguire praticamente per esercizio.

Il giudizio sul libro varia a seconda del punto di vista dal quale lo si considera. Al lettore salta subito agli occhi l'eccesso di voler spiegare da capo fino i concetti più elementari di ottica o di elettricità nei capitoli introduttivi delle parti seconda e terza, mentre poi il volume è diretto per esplicita dichiarazione dell'autore a studenti degli ultimi anni di chimica ed ai laureati in chimica che certe cose dovrebbero ben saperle!

Peraltro l'esposizione è assai chiara e precisa ed il testo assai accurato.

Come quadro d'insieme dei metodi strumentali usati nell'analisi chimica, il testo è completo ed utile soprattutto ai chimici in quanto riempie un vuoto abbastanza sentito nella letteratura didattica scientifica.

Infine per i fisici italiani impegnati in qualche modo nell'insegnamento universitario, può essere un utile invito a riflettere sulla opportunità di rivedere una parte dei programmi di esercizi di Fisica I e II generalmente impartiti ai chimici negli Istituti Universitari.

F. Gaeta

J. R. Reitz and F. J. Milford. *Foundation of Electromagnetic Theory*. Addison-Wesley Publ. Co., Inc. Reading, Mass., U.S.A., London, England (1960); $ 8.75.

Non si può dire che manchino oggi libri di testo sui fondamenti dell'elettricità e del magnetismo per corsi universitari a carattere introduttivo, in particolare nella letteratura americana. Ma il fatto stesso che se ne continuino a pubblicare e che ciascuno di essi avanzi qualche pretesa di originalità nell'impostazione, fa pensare che questo campo non sia ancora da considerare del tutto sistemato dal punto di vista didattico. Ciò si comprende, del resto, se si considera che la decisione relativamente recente di adottare il sistema Giorgi razionalizzato, in luogo dei molteplici ed ibridi sistemi in uso, ha fatto sorgere dei problemi didattici e concettuali ben più delicati di quanto non si sarebbe forse previsto per un semplice cambiamento di unità di misura.

Il volume che abbiamo sotto gli occhi si rivolge ad « advanced undergraduate students in science », per i quali è presupposta una conoscenza qualitativa della fisica atomica, ed una preparazione matematica che si può ritenere corrispondente a quella del nostro primo biennio. Il cap. I richiama, del resto, in forma accessibile anche a chi non ne avesse precedente nozione, le principali nozioni dell'algebra e dell'analisi vettoriale.

Il cap. II, sull'elettrostatica, contiene, oltre alle relazioni usuali, uno sviluppo del potenziale di una distribuzione generica di cariche in termini di multipoli. Il cap. III tratta principalmente delle soluzioni dell'equazione di Laplace con diversi sistemi di coordinate e diverse condizioni al contorno: fornendo degli esempi di applicazione dei metodi della fisica matematica classica. Una trattazione relativamente ampia del campo elettrostatico nei mezzi dielettrici, e dei problemi connessi, nei cap. IV, V e VI, trattazione basata sulle conoscenze attuali della costituzione dei dielettrici, prepara il terreno al successivo studio delle proprietà magnetiche della materia.

Il vettore induzione magnetica B viene definito, nel cap. VIII, per mezzo dell'espressione della forza di Lorentz su una carica in moto. Da essa si risale,

attraverso al teorema della circuitazione di Ampère, alla definizione di potenziale vettore. Dopo il cap. IX, sull'induzione elettromagnetica, seguono tre capitoli sulle proprietà magnetiche della materia, e problemi connessi. Il resto del volume tratta delle correnti variabili, equazioni di Maxwell ed applicazioni alla propagazione delle onde, per concludere con una breve introduzione allo studio dell'irraggiamento da una carica in moto accelerato. Un capitolo è pure inserito, che tratta, in modo necessariamente sommario, la fisica dei plasmi.

L'impostazione è nettamente matematica, senza riferimento diretto ad esperienze o dispositivi di misura. È fatto uso sistematico degli operatori e dei metodi dell'analisi vettoriale, in particolare per la dimostrazione dei teoremi. Le deduzioni sono per lo più complete e rigorose a partire dai principi fondamentali. La trattazione vettoriale consente tuttavia una notevole concisione senza scapito della chiarezza. Solo qua e là la preoccupazione della brevità ha indotto gli A. a contentarsi di affermazioni dogmatiche, o di cenni eccessivamente sommari: per es. nella trattazione del campo molecolare in un dielettrico, e più ancora in quella della conduzione metallica. Altri argomenti sono del tutto omessi: per es. i semiconduttori sono appena nominati, insieme con gli elettroliti.

In questo, come del resto anche in altri testi, la costante dielettrica del vuoto ε_0 viene introdotta, nel paragrafo sulla legge di Coulomb, attraverso il valore numerico approssimato $8.854 \cdot 10^{-12}$ coul$^2/n \cdot m^2$, senza indicare che ciò corrisponde a $10^7/4\pi c^2 = 1/\mu_0 c^2$. La relazione di ε_0 con la velocità della luce c non appare chiara neppure nell'appendice I, che pure si propone esplicitamente la « definizione logica delle unità del sistema MKS ». Essa è citata soltanto nel paragrafo sulla propagazione delle onde. Sembra allo scrivente che il prezzo che si paga nel sistema Giorgi con la introduzione delle costanti ε_0 e μ_0 sia giustificato soltanto quando si faccia apparire chiaramente il significato fisico di esse, come costanti che insieme condizionano la propagazione delle perturbazioni elettromagnetiche attraverso lo spazio vuoto.

Ma, del resto, la caratteristica di questo libro è proprio quella di mirare ad una esposizione concisa e rigorosa, soddisfacente dal punto di vista logico, più che a mettere in chiaro l'aspetto fisico delle questioni. Come già abbiamo detto, esso realizza bene, in massima, questo scopo. Una buona scelta di problemi, alla fine di ogni capitolo, aiuta ad approfondire gli argomenti trattati nel testo.

A. Rostagni

H. B. Callen – *Thermodynamics*, 376 pag., J. Wiley and Son, New York, 1960. $ 8.75

Professor Callen's textbook has been written as an introduction to thermodynamics for advanced undergraduate and first year graduate students. The mathematical equipment required is the standard elementary calculus plus a few special topics treated either in the text itself or in appendices.

The author has taken a postulational approach to his subject and gives the credit for its conception to Professor Lazslo Tisza. He adopts a set of four basic postulates concerning the existence of equilibrium states and the existence and properties of the entropy function, justifying them on the basis of intuition, experimental results and in an appendix, a qualitative statistical discussion. The mathematical development is then straightforward with an emphasis on theoretical principles to the exclusion of descriptions of apparatus and experimental detail.

Some special features of this presentation are the following: The differences and relations between extensive and intensive, dependant and independant variables, often a confusing affair for beginning students, are explained in a particularly clear manner. The method of Legendre transformations is used to introduce the thermodynamic potentials whose properties and special uses are well described. The extensive chapter on phase transitions is preceeded and well introduced by a chapter on the stability of thermodynamic systems.

The most interesting feature of this book, however, is the inclusion of several topics hardly touched on into the standard texts. These subjects are Tisza's theory of second order phase transitions, the theory of fluctuations, and irreversible thermodynamics. They are well handled in the limited space allowed and are very welcome in a basic thermodynamics course. Professor Callen does not present his own contributions to irreversible thermodynamics but for sake of brevity restricts the discussion to the Onsager reciprocity theorem. This is then applied in a succeeding chapter to the thermoelectric and thermomagnetic effects.

As an introduction to thermodynamics, however, this textbook has disadvatages as well as advantages. When the author gave up the traditional method of presentation of his subject he also abandoned to a large degree a coherent description of its historical development. References to the men who made the important contributions to thermodynamics and appreciations of their work are very brief and in some cases absent.

One other criticism concerns the chapters on applications. These are extremely condensed and though the usual formulae and approximations are derived, there are almost no comparisons with experimental results. The reasons for the success or failure of various treatments of the phenomena discussed and the important problems remaining are not mentioned; a fault, it seems to me, in a book otherwise so well designed to prepare and stimulate a student for research. This would have expanded the book somewhat beyond its present compact 370 pages but would certainly have been worthwhile.

To sum up: Professor Callen has written a modern textbook in thermodynamics for the serious student in physics. Because of its compactness and strict concentration on theoretical principles it makes strong demands on both the student and the teacher. In my opinion the author has been successful in his attempt to clarify both the underlying physical principles and the mathematical development of his subject by use of the postulational approach.

W. McCormick

Direttore responsabile: G. POLVANI Tipografia Compositori - Bologna

Questo fascicoloè stato licenziato dai torchi il 9-II-1961.